Erratum: Quantum measurements constrained by the third law of thermodynamics [Phys. Rev. A 107, 022406 (2023)]

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Item (iv) of Lemma B.2 is incorrect. Specifically, if a channel $\Phi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ maps all full-rank states to full-rank states, it may be the case that none of its fixed states $\rho_0 = \Phi(\rho_0)$ have full rank. Consider a channel defined as

$$\Phi(\cdot) := \lambda \mathbb{I}(\cdot) + (1-\lambda)\mathrm{tr}(\cdot)|\psi\rangle\langle\psi|, \tag{1}$$

where $0 < \lambda < 1$, $\mathbb{I}$ is the identity channel so that $\mathbb{I}(\rho) = \rho$ for all ρ, and $|\psi\rangle$ is a unit vector in $\mathcal{H}$. It is easily verified that $\Phi(\rho) \geqslant \lambda\, \rho$ for all ρ. Given that for any pair of positive operators A and B it holds that $A \geqslant B \Rightarrow \mathrm{rank}(A) \geqslant \mathrm{rank}(B)$, it follows that Φ is a rank nondecreasing channel, i.e., $\mathrm{rank}[\Phi(\rho)] \geqslant \mathrm{rank}(\rho)$ for all ρ. As such, Φ maps all full-rank states to full-rank states. However, now assume that $\Phi(\rho_0) = \rho_0$. It holds that $\rho_0 = \lambda\rho_0 + (1-\lambda)|\psi\rangle\langle\psi|$, which implies that $(1-\lambda)\rho_0 = (1-\lambda)|\psi\rangle\langle\psi|$, and so $\rho_0 = |\psi\rangle\langle\psi|$. In other words, the only fixed state of Φ is the pure state $|\psi\rangle\langle\psi|$. Indeed, one can also show that

$$\begin{aligned}
\Phi_{\mathrm{av}}(\cdot) &:= \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \Phi^n(\cdot) \\
&= \lim_{N\to\infty} \frac{1-\lambda^N}{N(1-\lambda)}\mathbb{I}(\cdot) + \left(1 - \frac{1-\lambda^N}{N(1-\lambda)}\right)\mathrm{tr}(\cdot)|\psi\rangle\langle\psi| \\
&= \mathrm{tr}(\cdot)|\psi\rangle\langle\psi|,
\end{aligned}$$

where we recall that Φ_{av} is a completely positive projection on the fixed points of Φ. It follows that while Φ maps all full-rank states to full-rank states, Φ does not have any full-rank fixed states.

As a consequence of the above-mentioned error, item (ii) of Lemma D.1 is also incorrect. More specifically, while a measurement scheme $\mathcal{M} := (\mathcal{H}_\mathcal{A}, \xi, \mathcal{E}, \mathsf{Z})$ that is constrained by the third law results in an E-channel $\mathcal{I}_\mathcal{X}$ which maps all full-rank states to full-rank states, it may be the case that the fixed-point set of $\mathcal{I}_\mathcal{X}$ does not contain any full-rank states. In other words, it may be the case that $\mathcal{I}_\mathcal{X}$ maps every full-rank state to some different full-rank state. In such a case, the results in Appendix E may not necessarily hold. Consequently, while Theorems 4.3 and 4.5 are independent of the fixed-point structure of $\mathcal{I}_\mathcal{X}$ and so are left unchanged, it may be the case that Theorems 4.1 (as well as its corollaries in Proposition F.2), 4.2, and 4.4 are not valid in general. Specifically, if it is possible for a third-law constrained measurement to implement a channel $\mathcal{I}_\mathcal{X}$ that perturbs all full-rank states, then it may still be possible for such a measurement of an unsharp—but not necessarily completely unsharp—observable to not disturb any observable that commutes with it (Theorem 4.1), to not disturb itself (Theorem 4.2), or to be ideal (Theorem 4.4). Note that the impossibility of nondisturbance and ideality is still guaranteed for the class of sharp observables; if an observable is sharp, then its measurement is of the first kind (does not disturb itself) if and only if it is repeatable, which is ruled out by Theorem 4.3. Since sharp observables are commutative, this immediately implies that a third-law constrained measurement of a sharp observable will disturb at least some observable that commutes with it. Finally, for sharp observables, the ideal measurements are precisely the Lüders instruments. However, by Proposition D.1, the third law rules out Lüders instruments for any observable that is not completely unsharp.

The universal validity of the aforementioned theorems is guaranteed to hold under the added assumption that there exists a full-rank fixed state $\rho_0 = \mathcal{I}_\mathcal{X}(\rho_0)$. There are special cases where an E-channel $\mathcal{I}_\mathcal{X}$ that is constrained by the third law is guaranteed to not perturb at least one full-rank state, as shown by the following examples.

Example 1: Rank-1 observables. Consider the case where E is a rank-1 observable, i.e., where every effect E_x is proportional to a rank-1 projection $P_x \equiv |\psi_x\rangle\langle\psi_x|$, where $\{|\psi_x\rangle\}$ are not necessarily mutually orthogonal. Since E is an observable, then for every state ρ it holds that $\mathrm{tr}(\mathsf{E}_x\rho) > 0$ for some x; however, $\mathrm{tr}(\mathsf{E}_x\rho) = 0 \iff P_x\rho P_x = \mathbb{O}$. As such, for every state ρ, it will hold that $P_x\rho P_x$ will have full rank in the 1-dimensional subspace $P_x\mathcal{H}_\mathcal{S}$ for at least some x. Now assume that the measurement of a rank-1 observable is constrained by the third law. By item (iv) of Lemma D.1 it follows that for every state ρ, $\mathcal{I}_x(\rho)$ has full rank for some x. As such, any fixed state $\rho_0 = \mathcal{I}_\mathcal{X}(\rho_0) = \sum_x \mathcal{I}_x(\rho_0)$ must have full rank.

Example 2: Thermodynamically free measurements. Consider the case where the compound of the system and apparatus has an additive Hamiltonian $H = H_\mathcal{S} \otimes \mathbb{1}_\mathcal{A} + \mathbb{1}_\mathcal{S} \otimes H_\mathcal{A}$, so that for some inverse temperature $\beta > 0$ the Gibbs state of the compound system is $e^{-\beta H}/\mathrm{tr}(e^{-\beta H}) = \rho_\beta \otimes \xi_\beta$, where $\rho_\beta := e^{-\beta H_S}/\mathrm{tr}(e^{-\beta H_S})$ and $\xi_\beta := e^{-\beta H_\mathcal{A}}/\mathrm{tr}(e^{-\beta H_\mathcal{A}})$ are the Gibbs states of the system and apparatus, respectively. Note that such states have full rank. If the apparatus is prepared in ξ_β and if the measurement interaction channel $\mathcal{E}$ is Gibbs preserving, then the measurement will be constrained by the third law. However, in such a case it will hold that $\mathcal{I}_\mathcal{X}(\rho_\beta) = \mathrm{tr}_\mathcal{A}[\mathcal{E}(\rho_\beta \otimes \xi_\beta)] = \mathrm{tr}_\mathcal{A}(\rho_\beta \otimes \xi_\beta) = \rho_\beta$, and so $\mathcal{I}_\mathcal{X}$ will have a full-rank fixed state. In particular, the thermal instruments introduced in [1] are constrained by the third law and are guaranteed to have a full-rank fixed state.

Example 3: Measurement channels with bounded smallest eigenvalues. Assume that the measurement channel $\mathcal{I}_\mathcal{X}$ is such that if the smallest eigenvalue of an arbitrary state ρ is larger than $\epsilon > 0$, then the smallest eigenvalue of $\mathcal{I}_\mathcal{X}(\rho)$ will also be larger than ϵ. This is a stronger condition than merely mapping all full-rank states to full-rank states. For example, consider the channel Φ defined in Eq. (1) and let ρ be a state whose smallest eigenvalue is ϵ, with the corresponding eigenvector orthogonal to $|\psi\rangle$. In such a case, the smallest eigenvalue of $\Phi(\rho)$ will be $\lambda\epsilon < \epsilon$. In other words, repeated applications of Φ will continuously reduce the smallest eigenvalue of the input state, so in the limit of infinite applications, such a state will be taken to the pure state $|\psi\rangle\langle\psi|$. However, if the measurement channel $\mathcal{I}_\mathcal{X}$ satisfies the aforementioned property, then even in the limit of infinite applications, the smallest eigenvalue of the output will be larger than ϵ if the smallest eigenvalue of the input is also larger than ϵ. In such a case, and in contrast to the incorrect proof of item (iv) of Lemma B.2, it will hold that $\mathcal{I}_{\mathrm{av}}(\rho) := \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}(\mathcal{I}_\mathcal{X})^n(\rho)$ will have full rank, so $\mathcal{I}_\mathcal{X}$ will have a full-rank fixed state.

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[1] M. H. Mohammady, Thermodynamically free quantum measurements, J. Phys. A: Math. Theor. **55**, 505304 (2022).