Entangled graphs: Bipartite entanglement in multiqubit systems

Martin Plesch[1] and Vladimír Bužek[1,2]
[1]*Research Center for Quantum Information, Slovak Academy of Sciences, Dúbravská cesta 9, Bratislava, Slovakia*
[2]*Department of Mathematical Physics, National University of Ireland, Maynooth, Co. Kildare, Ireland*
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Quantum entanglement in multipartite systems cannot be shared freely. In order to illuminate basic rules of entanglement sharing between qubits, we introduce a concept of an entangled structure (graph) such that each qubit of a multipartite system is associated with a point (vertex), while a bipartite entanglement between two specific qubits is represented by a connection (edge) between these points. We prove that any such entangled structure can be associated with a *pure* state of a multiqubit system. Moreover, we show that a pure state corresponding to a given entangled structure is a superposition of vectors from a subspace of the 2^N-dimensional Hilbert space, whose dimension grows *linearly* with the number of entangled pairs.

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I. INTRODUCTION

The entanglement is a key ingredient of quantum mechanics [1,2]. In the last decade, it has been identified as a key resource for quantum information processing. In particular, quantum computation [3,4], quantum teleportation [5], quantum dense coding [6], certain types of quantum key distributions [7], and quantum secret sharing protocols [8] are based on the existence of entangled states.

The nature of quantum entanglement between two qubits is well understood by now. In particular, the necessary and sufficient condition for inseparability of two-qubit systems has been derived by Peres [9] and Horodecki *et al.* [10]. Reliable measures of bipartite entanglement have been introduced and well analyzed (see for instance Refs. [11,12]). On the other hand, it is a very difficult task to generalize the analysis of entanglement from two to multipartite systems. The multipartite entanglement is a complex phenomenon. One of the reasons is that quantum entanglement cannot be shared freely among many particles. For instance, having four qubits, we are able to prepare a state with two *e*-bits (two Bell pairs, as an example), but not more. This means that the structure of quantum mechanics imposes strict bounds on bipartite entanglement in multipartite systems. This issue has been first addressed by Wootters *et al.* [13,14] who have derived important bounds on shared bipartite entanglement in multiqubit systems. In fact, one can solve a variational problem to answer a question: What is a pure multipartite state with specific constraints on bipartite entanglement? O'Connors and Wootters [14] have studied what is the state of a multiqubit ring with maximal possible entanglement between neighboring qubits. Another version of the same problem has been analyzed by Koashi *et al.* [15] who have derived an explicit expression for the multiqubit completely symmetric state (entangled web) in which all possible pairs of qubits are maximally entangled.

In his recent work, Dür [17] has introduced a concept of *entanglement molecules*. Dür has shown that an arbitrary entanglement molecule can be represented by a *mixed* state of a multiqubit system. On the other hand, in his work the problem of pure multipartite states with specific entangled pairs of qubits has not been discussed thoroughly. Specifically, Dür has considered just the condition of inseparability for given set of pairs, but he did not impose a strict condition of separability for the remaining pairs of qubits.

Following these ideas we analyze in the present paper an object, entangled structure, which will be called through the paper as the *entangled graph* [16]. In the graph, each qubit is represented as a vertex and an edge between two vertices denotes entanglement between these two particles (specifically, the corresponding two-qubit density operator is inseparable). The central issue of the paper is to show that any entangled graph with N vertices and k edges can be associated with a *pure* multiqubit state. We prove this result constructively, by showing the explicit expression of corresponding pure states. We show that any entangled graph of N qubits can be represented by a pure state from a subspace of the whole 2^N-dimensional Hilbert space of N qubits. The dimension of this subspace is at most quadratic in number of qubits.

II. SIMPLE EXAMPLE

In general, an N-partite system can exhibit various types of multipartite correlations, ranging from bipartite entanglement to intrinsic multipartite correlations of the Greenberger-Horne-Zeilinger (GHZ) nature. Correlations associated with the system specify its state (certainly, this specification is not necessarily unique). Ideally, we would like to know the whole hierarchy of quantum correlations in the multipartite system. We are able to determine and quantify bipartite quantum correlations. Unfortunately, for existence of intrinsic N-qubit correlations we even do not have sufficient and necessary conditions (see Refs. [12,13]). Nevertheless, as suggested by Coffman, Kundu, and Wootters (CKW) [13] it is very instructive to understand how a bipartite entanglement is "distributed" in N-qubit system. The inequalities derived by these authors (the so-called CKW inequalities) open new possibilities how to understand the complex problem of bounds on shared entanglement. The CKW inequalities utilize the measure of entanglement called concurrence as introduced by Wootters *et al.* [18]. This measure is defined as follows: Let us assume a two-qubit system prepared in the state described by the density operator ρ. From this operator

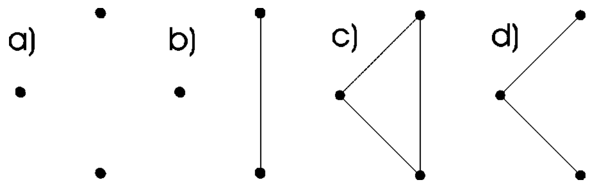

FIG. 1. Four different classes of entangled graphs associated with states of three qubits: (a) separable and GHZ-like states with no bipartite entanglement, (b) Bell-like states, (c) *W*-like states, and (d) a new category of entangled states.

one can evaluate the so-called spin-flipped operator defined as

$$\tilde{\rho}=(\sigma_y\otimes\sigma_y)\rho^*(\sigma_y\otimes\sigma_y), \tag{2.1}$$

where σ_y is the Pauli matrix and a star denotes a complex conjugation. Now, we define the matrix

$$R=\rho\tilde{\rho}, \tag{2.2}$$

and label its (non-negative) eigenvalues, in decreasing order $\lambda_1,\lambda_2,\lambda_3$, and λ_4. The concurrence is then defined as

$$C=\max\{0,\sqrt{\lambda_1}-\sqrt{\lambda_2}-\sqrt{\lambda_3}-\sqrt{\lambda_4}\}. \tag{2.3}$$

This function serves as an indicator whether the two-qubit system is separable (in this case $C=0$), while for $C>0$ it measures the amount of bipartite entanglement between two qubits with a number between 0 and 1. Larger the value of C stronger the entanglement between two qubits is.

Unfortunately, no simple measures of entanglement are known for multiqubit systems. Nevertheless, it is still of importance to understand how a bipartite entanglement is distributed in *N*-qubit system. In this paper, we will utilize a concept of entangled graph to illuminate some aspects of the problem.

Using the concurrence, we can easily associate an entangled graph with every *N*-partite state. On the other hand, there is no one to one correspondence between graphs and states. For instance all separable states with N qubits have the same graph—N vertices and no edges. Also all GHZ like states for $N>2$ would have the same graph. The question we are going to address can be formulated as follows: Is it possible to construct at least one *pure* state for a given graph?

We start our discussion with the simplest nontrivial example: Let us consider three qubits. Pure states of three qubits can be divided into six classes (see Ref. [19]): separable states, bipartite entangled states (three classes respective to the permutation), the *W*-type states, and the GHZ-type states. On the other hand in Fig. 1, we represent all possible graphs for a three-partite system. Separable and GHZ-like states correspond to the case (a), bipartite entangled states are represented by the graph (b), while the *W*-type states are represented by the graph (c). Obviously, one can imagine also an additional type of a graph, when a given qubit (labeled as the qubit 2) is entangled with two others (labeled as 1 and 3, respectively), while the qubits 1 and 3 are not entangled. The question is whether this type of a graph [see Fig. 1(d)] can exist. Does the entanglement between qubits 1 and 2, and 2 and 3 induce the entanglement between qubits 1 and 3?

In order to illuminate this simple problem let us first consider mixed states. According to Dür (see Ref. [17]) *mixed* states associated with the graph (d) do exist and have the form

$$\rho=a|\Psi^+\rangle_{12}\langle\Psi^+|\otimes|0\rangle_3\langle 0|+(1-a)|\Psi^+\rangle_{23}\langle\Psi^+|\otimes|0\rangle_1\langle 0|, \tag{2.4}$$

where $|\Psi^+\rangle=(|01\rangle+|10\rangle)/\sqrt{2}$ is a Bell state. By inspection one can check that for $0<a<1$ the mixed state (2.4) exhibits required correlations.

From the explicit expression for the mixed state (2.4) associated with the graph (d), we might try to express a pure state corresponding to the same graph as follows:

$$|\Phi\rangle=\alpha|\Psi^+\rangle_{12}|0\rangle_3+\sqrt{1-\alpha^2}|\Psi^+\rangle_{23}|0\rangle_1. \tag{2.5}$$

This state has required properties imposed on correlations betweem qubits 1-2 and 1-3. But, it also exhibits entanglement between qubits 2 and 3, and therefore the corresponding graph is of the type (c). In order to find a pure state for the graph (d), we will consider a whole family of states of the form

$$|\Phi\rangle=\alpha|000\rangle+\beta|100\rangle+\gamma|110\rangle+\sqrt{1-\alpha^2-\beta^2-\gamma^2}|111\rangle. \tag{2.6}$$

It is straightforward to calculate concurrencies for all pairs of qubits in the state (2.6). We find that $C(1,3)$ is always zero, while $C(1,2)$ and $C(2,3)$ are nonzero for all nonzero values of involved probability amplitudes. By inspection it is possible to determine that the state (2.6) belongs to the class of GHZ states, since it contains intrinsic three-partite entanglement.

We can conclude this simple example by saying that all three-qubit entangled graphs can be realized by pure three-qubit states. We note that the classification of states according to entangled graphs, representing the two-partite entanglement is incompatible with the classification presented in Ref. [19]. We see that two types of states (GHZ states and separable states) have the same graph. On the other hand, two states of the same class (GHZ states) can be represented by different graphs.

III. *N*-PARTICLE SYSTEM

Let us first consider entangled graphs associated with mixed *N*-qubit states. These graphs consist of N vertices. Let the parameter k denote the number of edges in the graph, with the condition

$$0\leqslant k\leqslant\frac{N(N-1)}{2}. \tag{3.1}$$

Then let us define a set S with k members. These will be pairs of qubits between which we expect entanglement; thus, for every $i<j$,

$$\{i,j\}\in S \quad\Leftrightarrow\quad C(i,j)>0,$$

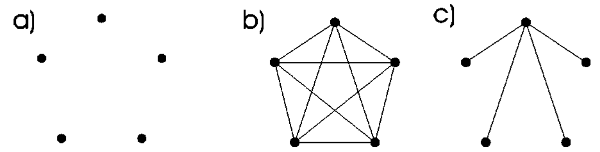

FIG. 2. Examples of entangled graphs associated with states of five qubits: (a) separable states, or any other states with no bipartite entanglement, (b) W-type states, and (c) star-shaped states.

$$\{i,j\} \notin S \quad \Leftrightarrow \quad C(i,j)=0. \tag{3.2}$$

A state of the form

$$|\Psi\rangle_{ij}=|\Psi^{+}\rangle_{ij}|0\cdots 0\rangle_{\overline{ij}} \tag{3.3}$$

exhibits entanglement between qubits i and j and nowhere else. In Eq. (3.3) the vector $|\Psi^{+}\rangle_{ij}=(|01\rangle+|10\rangle)/\sqrt{2}$ represents the maximally entangled Bell state between pairs of two qubits i and j. The rest of $N-2$ qubits are assumed to be in the product state $|0\cdots 0\rangle_{\overline{ij}}$. Dür in Ref. [17] has proposed a *mixed* state of N qubits, which corresponds to a graph defined by the set S in the form

$$\rho=\frac{1}{k}\sum_{\{i,j\}\in S}|\Psi\rangle_{ij}\langle\Psi|_{ij}. \tag{3.4}$$

It is much more complex task to find a pure state of N qubits corresponding to a specific graph. We will solve this problem below.

Pure states

We start our analysis with entangled graphs that exhibit specific symmetries. Certainly the two most symmetric graphs are those representing separable states [no edges—see Fig. 2(a)] and those representing W states, with all vertices connected by edges. A representative of a pure completely separable state is described by the vector $|\Psi\rangle=|0\cdots 0\rangle$. The W state $|W\rangle_N=1/\sqrt{N}|N-1,1\rangle$, is a maximally symmetric state with one qubit in state $|1\rangle$ and $N-1$ qubits in state $|0\rangle$ (see Refs. [15,17]). This state maximizes the bipartite concurrence—its value is given by the expression $C=2/\sqrt{N}$. We see that the most symmetric entangled graphs do correspond to specific pure multiqubit states.

Let us now consider graphs with a lower symmetry. For instance, a star-shaped graph [Fig. 2(c)]. In this case, the given qubit is (equally) entangled with all other qubits in the system, that in turn are not entangled with any other qubit.

Dür [17] has proposed an explicit expression for a pure state associated with this type of entangled graph

$$|\Psi\rangle=\frac{1}{\sqrt{2}}|1\rangle|0\cdots 0\rangle+\frac{1}{\sqrt{2}}|0\rangle|N-2,1\rangle. \tag{3.5}$$

In fact, this state maximizes the concurrence between the first and any other qubit. But, the remaining qubits are still mutually entangled. So the state (3.5) is represented by the graph (b) (all vertices are connected) rather than graph (c). In our analysis, we require more stringent constraints than in Ref. [17], where only the conditions on the presence of entanglement between specific qubits have been imposed. We require the conditions (3.2), that is, the presence and absence of bipartite entanglement for given pairs of qubits in the system.

We find that a pure state which indeed is represented by the star-shaped graph [see Fig. 2(c)] is given by the expression

$$|\Psi\rangle=\alpha|W\rangle_N+\beta|0\rangle|1\cdots 1\rangle, \tag{3.6}$$

with the normalization condition $|\alpha|^2+|\beta|^2=1$. For $N>4$, the reduced two-qubit density operator for the first and any other qubit in the system reads

$$\rho_{1i}=\begin{pmatrix} \frac{N-2}{N}|\alpha|^2 & 0 & 0 & 0\\ 0 & |\alpha|^2\frac{1}{N}+|\beta|^2 & |\alpha|^2\frac{1}{N} & 0\\ 0 & |\alpha|^2\frac{1}{N} & |\alpha|^2\frac{1}{N} & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.7}$$

One of the eigenvalues of the partially transformed matrix

$$\lambda=|\alpha|^2\frac{n-2-\sqrt{n^2+8-4n}}{n} \tag{3.8}$$

is negative for every $\alpha>0$. Consequently, using the Peres-Horodecki criterion we see that the first qubit is indeed entangled with any other qubit in the system for any nontrivial value of α. Now we have to show, that all other qubits in the system are not mutually entangled (i.e., all pairs of qubits $\{i,j\}$, where $1<i<j<N$ are separable). The reduced density operator describing a state of qubits i and j reads

$$\rho_{ij}=\begin{pmatrix} \frac{N-2}{N}|\alpha|^2 & 0 & 0 & 0\\ 0 & |\alpha|^2\frac{1}{N} & |\alpha|^2\frac{1}{N} & 0\\ 0 & |\alpha|^2\frac{1}{N} & |\alpha|^2\frac{1}{N} & 0\\ 0 & 0 & 0 & |\beta|^2 \end{pmatrix}. \tag{3.9}$$

The smallest eigenvalue of the partially transposed operator is

$$\lambda=\frac{N-2|\alpha|^2-\sqrt{\delta}}{2N}, \tag{3.10}$$

where

$$\delta=N^2-4|\alpha|^2(N-1)N+4|\alpha|^4[2+(N-2)N].$$

We see that for all α such that

$$|\alpha| \leqslant \frac{\sqrt{N^2-2N}}{N-1}, \tag{3.11}$$

the smallest eigenvalue λ is non-negative. Consequently, the corresponding density operator is separable. Thus, we have found a family of states that correspond to the desired graph. For the special case of $N=4$, the reduced operators have a form different from Eqs. (3.7) and (3.9) and also the final condition is more complicated. However, it is quite easy to find one example of the state of four qubits corresponding to the star-shaped graph. The state vector reads

$$\begin{aligned}|\Psi\rangle &= \frac{1}{\sqrt{5}}(|0111\rangle+|0001\rangle+|0010\rangle+|0100\rangle+|1000\rangle)\\ &= \frac{2}{\sqrt{5}}|W\rangle_4+\frac{1}{\sqrt{5}}|0111\rangle.\end{aligned} \tag{3.12}$$

Above, we have analyzed the most symmetric entangled graphs. In what follows, we propose a general algorithm how to construct a pure state for an arbitrary graph. Let us consider a pure state of N ($N>4$) qubits described by the vector

$$|\Psi\rangle = \alpha|0\cdots 0\rangle+\beta|1\cdots 1\rangle+\sum_{\{i,j\}\in S}\frac{\gamma}{\sqrt{k}}|1\rangle_i|1\rangle_j|0\cdots 0\rangle_{\overline{ij}}, \tag{3.13}$$

with the normalization condition $|\alpha|^2+|\beta|^2+|\gamma|^2=1$. In what follows, we will show that for a certain range of parameters this state matches a graph given by the condition (3.2).

First, we show that a pair of qubits i and j such that $\{i,j\}\in S$ is indeed entangled. The corresponding reduced density operator reads

$$\rho_{ij} = \begin{pmatrix} |\alpha|^2+|\gamma|^2\frac{k-n_i-n_j+1}{k} & 0 & 0 & \frac{\alpha\gamma^*}{\sqrt{k}} \\ 0 & |\gamma|^2\frac{n_i-1}{k} & |\gamma|^2\frac{n_{ij}}{k} & 0 \\ 0 & |\gamma|^2\frac{n_{ij}}{k} & |\gamma|^2\frac{n_j-1}{k} & 0 \\ \frac{\alpha^*\gamma}{\sqrt{k}} & 0 & 0 & |\beta|^2+\frac{|\gamma|^2}{k} \end{pmatrix}, \tag{3.14}$$

where n_i is the number of connections originating from the ith vertex (the number of qubits we wish to have entangled with the ith one) and n_{ij} is the number of vertices that are connected directly with the ith and jth vertex. The following inequalities for these variables hold:

$$\begin{aligned} &1 \leqslant n_i \leqslant k,\\ &0 \leqslant n_{ij} < \frac{k}{2},\\ &2 \leqslant n_i+n_j \leqslant k+1,\\ &1 \leqslant n_i n_j \leqslant \frac{(k+1)^2}{4}. \end{aligned} \tag{3.15}$$

One of the eigenvalues of the density operator obtained by the partial transposition of the operator (3.14) reads

$$\lambda = \frac{|\gamma|}{2k}\left[|\gamma|(n_i+n_j-2)-\sqrt{4|\alpha|^2k+|\gamma|^2(n_i-n_j)^2}\right].$$

In the nontrivial case of $|\gamma|>0$, we need only to show that

$$|\gamma|^2(n_i+n_j-2)^2 < 4|\alpha|^2k+|\gamma|^2(n_i-n_j)^2. \tag{3.16}$$

If we use the inequalities (3.15), we find the following constraints:

$$\begin{aligned} &|\gamma|^2(n_i+n_j-2)^2 < |\gamma|^2k^2 \leqslant 4|\alpha|^2k,\\ &4|\alpha|^2k \leqslant 4|\alpha|^2k+|\gamma|^2(n_i-n_j)^2, \end{aligned} \tag{3.17}$$

from which it follows that if the condition

$$0 < |\gamma|^2k \leqslant 4|\alpha|^2 \tag{3.18}$$

is fulfilled then a specific pair qubits described by the density operator (3.14) is entangled.

Till now, we have proved that a specific pair of qubits in multipartite system is entangled. In order, to show that the corresponding state vector indeed is associated with a desired entangled graph, we have to show that all other pairs of qubits are separable. Density operators for pairs of qubits $\{i,j\}\notin S$ are given by the expression

$$\rho_{ij}=\begin{pmatrix} |\alpha|^2+|\gamma|^2\frac{k-n_i-n_j}{k} & 0 & 0 & 0 \\ 0 & |\gamma|^2\frac{n_i}{k} & |\gamma|^2\frac{n_{ij}}{k} & 0 \\ 0 & |\gamma|^2\frac{n_{ij}}{k} & |\gamma|^2\frac{n_j}{k} & 0 \\ 0 & 0 & 0 & |\beta|^2 \end{pmatrix}, \tag{3.19}$$

with the involved parameters satisfying the set of inequalities

$$0\leqslant n_i \leqslant k,$$
$$0\leqslant n_{ij}\leqslant \frac{k}{2},$$
$$0\leqslant n_i+n_j\leqslant k,$$
$$0\leqslant n_i n_j \leqslant \frac{k^2}{4}. \tag{3.20}$$

Instead of checking that all the eigenvalues of the corresponding partially transposed operator are non-negative, we will show that under certain conditions the concurrence of the state (3.20) will be zero. The eigenvalues of the operator R given by Eq. (2.2) are

$$\lambda_1=\lambda_2=|\alpha\beta|^2+|\gamma|^2\frac{k-n_i-n_j}{k},$$

$$\lambda_{3,4}=|\gamma|^4\left(\frac{n_{ij}\pm\sqrt{n_i n_j}}{k}\right)^2, \quad \lambda_4\geqslant\lambda_3, \tag{3.21}$$

and according to the definition of the concurrence (2.3), it is enough to show that $\lambda_1\geqslant\lambda_4$ (since then λ_1 is the maximal eigenvalue and already $\sqrt{\lambda_1}-\sqrt{\lambda_2}=0$ and so the concurrence vanishes). That is, we require that

$$|\alpha\beta|^2+|\gamma|^2\frac{k-n_i-n_j}{k}\geqslant|\gamma|^4\left(\frac{n_{ij}+\sqrt{n_i n_j}}{k}\right)^2. \tag{3.22}$$

When we use the inequalities (3.20), we obtain the final condition

$$|\alpha\beta|^2\geqslant|\gamma|^4>0, \tag{3.23}$$

which guarantees that the state (3.19) is separable.

One can check that there are many states which fulfill the conditions (3.18) and (3.23). In particular, let us assume the state (3.13) with

$$\alpha=\frac{k}{\sqrt{k^2+2k+4}},$$

$$\beta=\frac{2\alpha}{k},$$

$$\gamma=\alpha\sqrt{\frac{2}{k}}. \tag{3.24}$$

This state indeed corresponds to the desired graph. This proves that one can associate with an arbitrary entangled graph a pure state. Moreover, by construction we have proved that, in general, this state is a superposition of at most N^2 vectors from the 2^N-dimensional Hilbert space of N qubits.

IV. CONCLUSION

We have introduced a concept of the entangled graphs, that is, an entangled multiqubit structure such that every qubit is represented by a vertex, while entanglement between two qubits is represented as an edge between relevant vertices. We have shown that for every possible graph with nonweighted (see below) edges there exists a pure state, which represents the graph. Moreover, such state can be constructed, as a superposition of small number of states from a subspace of the Hilbert space. The dimension of this subspace grows linearly with the number of entangled pairs (thus, in the worst case, quadratically with the number of particles).

It is clear that introducing a ''weight'' to the edges of the graphs would lead to new interesting questions. The weight should correspond to a value of the concurrence between the two qubits that are connected by the given edge. Dür in Ref. [17] has addressed this question briefly in the context of mixed states associated with entanglement molecules. As shown in our paper, the issue of shared bipartite entanglement in *pure* multiqubit systems is much more complex issue. Nevertheless, it is of great interest to find out some general bounds on the amount of possible shared bipartite entanglement in a given entangled graph. Another problem directly related to the issue of weighted graphs is how to maximize bipartite entanglement for given graphs. There, for every graph one could find the optimal state with maximal concurrencies on defined pairs of qubits, as it was made for specific cases in Refs. [14,15].

States with defined bipartite entanglement properties are of a possible practical use: In communications protocols, like quantum secret sharing [8] or quantum oblivious transfer [20] one needs many-particle states with specific bipartite entanglement properties. Therefore, deep understanding of possible entangled graphs can help us to understand structure of quantum correlation and the corresponding bounds on quantum communications and quantum information processing.

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