Quantum-controlled measurement device for quantum-state discrimination

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We propose a ‘‘programmable’’ quantum device that is able to perform a specific generalized measurement from a certain set of measurements depending on a quantum state of a ‘‘program register.’’ In particular, we study a situation when the programmable measurement device serves for the unambiguous discrimination between nonorthogonal states. The particular pair of states that can be unambiguously discriminated is specified by the state of a program qubit. The probability of successful discrimination is not optimal for all admissible pairs. However, for some subsets it can be very close to the optimal value.

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I. INTRODUCTION

Quantum measurements are inevitable parts of all quantum devices. Specific sets of quantum measurements are essential for optimal quantum-state estimation [1,2]. Quantum measurements also represent the final step of any quantum computation [3]. In many situations the choice of an optimal measurement depends on the task to be performed. For instance, in the case of quantum-state discrimination the choice of the measurement is given by the specific pair of states that are supposed to be discriminated.

A selection of a specific measurement can be performed on a ‘‘classical’’ level. That is, the parameters of the measurement (e.g., the orientation of the Stern-Gerlach apparatus) are completely described classically. On the other hand the parameters determining the character of quantum measurement can be encoded in a state of a quantum ‘‘program’’ register. Certainly, in this situation one could perform a measurement on a program register and estimate the parameters specifying the measurement to be performed on the system. With these parameters one can then ‘‘classically’’ adjust the measurement apparatus and perform the measurement over the system. The other option is that the quantum program register directly determines the measurement to be performed on the system. This purely quantum control can be realized *without* an intermediate intervention of an observer.

Therefore it is interesting to understand whether it is possible to construct a universal (multipurpose) quantum-measurement device (‘‘quantum multimeter’’). That is, an apparatus that could perform a specific class of generalized measurements [positive operator valued measure (POVM)] in such a way that each member of this class could be selected by a particular quantum state of a ‘‘program register.’’ The key property of this approach is a possibility to control the choice of the measurement (e.g., the measurement basis in case of a projective measurement) by a (in principle, unknown) quantum state of the program register. This state can be determined, for instance, as a result of some quantum-information process.

The generalized measurement is defined by the fact that the probability of each of its results (the number of results may be, in general, larger than the dimension of the Hilbert space of the measured system) is given by the expression $p_\mu = \mathrm{Tr}_\mathrm{S}(\mathsf{A}_\mu \rho_\mathrm{S})$, where ρ_S is the state of the system and A_μ are *positive* operators that constitute the decomposition of the identity operator ($\sum_\mu \mathsf{A}_\mu = \mathbb{1}$). This is the reason why it is called positive operator valued measure [1,2,4]. Each POVM can be implemented using an ancillary quantum system in a specific state and realizing a projective von Neumann measurement on the composite system [5]. In other words, if one has an ‘‘input’’ (measured) state ρ_S in the Hilbert space $\mathcal{H}_\mathrm{S}$ it is always possible to find some state ρ_A in a space $\mathcal{H}_\mathrm{A}$ and a set of orthogonal projectors $\{\mathsf{E}_\mu\}$ acting on $\mathcal{H}_\mathrm{S} \otimes \mathcal{H}_\mathrm{A}$ ($\sum_\mu \mathsf{E}_\mu = \mathbb{1}$) such that

$$\mathsf{A}_\mu = \mathrm{Tr}_\mathrm{A}(\mathsf{E}_\mu \rho_\mathrm{A}) \tag{1}$$

are positive operators as discussed above.

In general, we can assume that the initial state of the ancilla can be prepared with an arbitrary precision. The ancilla can be considered as a part of the ‘‘program register.’’ Further, we note that the general projection measurement on the composite system can be represented by a unitary transformation on the composite system followed by a fixed projection measurement (e.g., independent projective measurements on individual qubits). Therefore the problem of designing the programmable quantum multimeter reduces to the question of whether an arbitrary unitary operation (on the Hilbert space with a given dimension) can be encoded in some quantum state of a program register of a finite dimension. It was shown that the answer to this question is ‘‘No.’’ Nielsen and Chuang proved that any two inequivalent operations require orthogonal program states [6]. Thus the number of encoded operations cannot be higher than the dimension of the Hilbert space of the program register. Since, in general, the set of all unitary operations can be infinite, the result of Nielsen and Chuang implies that no universal programmable gate array can be constructed using finite resources. They showed, however, that if the gate array is probabilistic, a universal gate array is possible. A probabilistic array is one that requires a measurement to be made at the output of the program register, and the output of the data register is only accepted if a particular result, or set of results, is obtained. This will happen with a probability, which is less than one.

Vidal and Cirac [7] have presented a probabilistic programmable quantum gate array with a finite program register which can realize a family of operations with one continuous parameter. Recently, Hillery *et al.* [8] have proposed a more general quantum processor that can perform probabilistically any operation (not only unitary) on a qubit. Another aspect of encoding quantum operations in states of a program register has been discussed by Huelga *et al.* [9]. They dealt with the so-called teleportation of unitary operations. Unfortunately, the probabilistic realization of unitary operations cannot help to build a programmable quantum multimeter in the way mentioned above. The reason is that the probabilistic implementation of a given operation leads, at the end, to a *different* POVM than the deterministic implementation of the same operation would lead to. (The new $N+1$ component POVM with one more output corresponding to a ''failure'' is different from the desired N component one. For example, if the desired POVM already contains an inconclusive output then if it is implemented probabilistically the total probability of the ''failure'' increases in general.)

In general, we can describe a quantum multimeter as a (fixed) unitary operation acting on the measured system (or a ''data register'') and an ancillary system (''program register'') together and a (fixed) projective measurement realized afterwards on the same composite system. Clearly, such a device can perform only a restricted set of POVMs. One can, therefore, ask what is the optimal unitary transformation that enables us to implement ''the largest set of POVMs'' (in comparison with the set of POVMs that would be obtainable when we allowed any unitary transformation on the same Hilbert space). One can also ask what unitary transformation can help to approximate all the POVMs (generated by an arbitrary unitary transformation) with the highest precision (fidelity) on average. Clearly, the last task requires definition of the distance measure between two POVMs. This is an interesting problem *per se*, however, it goes far beyond the scope of our considerations here. Both optimization problems mentioned above are rather nontrivial. Moreover, the introduced scheme is perhaps too general from a practical point of view. Therefore in the present paper we will concentrate our attention on a more specific case: On the problem of state discrimination.

We stress once again that a quantum multimeter as discussed in the present paper is a device which, in contrast to its classical counterpart, is controlled (switched, programmed) by the quantum states of a program register that are allowed to be mutually nonorthogonal.

II. DISCRIMINATION OF QUANTUM STATES

In the following we will study a particular example of a ''quantum multimeter'' serving for a programmable unambiguous state discrimination. So, it is in place to say a few words about quantum-state discrimination now.

A general *unknown* quantum state cannot be determined completely by a measurement performed on a single copy of the system. But the situation is different if *a priori* knowledge is available [1,2,4], e.g., if one works only with states from a certain discrete set. Even quantum states that are mutually nonorthogonal can be distinguished with a certain probability provided they are linearly independent (for a review see Ref. [10]). There are, in fact, two different optimal strategies [11]: First, the strategy that determines the state with the minimum probability for the error [1,2] and, second, unambiguous or error-free discrimination (the measurement result never wrongly identifies a state) that allows the possibility of an inconclusive result (with a minimal probability in the optimal case) [12–16]. We will concentrate our attention to the unambiguous state discrimination. It has been first investigated by Ivanovic [12] for the case of two equally probable nonorthogonal states. Peres [14] solved the problem of discrimination of two states in a formulation with POVM measurement. Later Jaeger and Shimony [15] extended the solution to arbitrary *a priori* probabilities. Chefles and Barnett [16] have generalized Peres's solution to an arbitrary number of equally probable states which are related by a symmetry transformation. Unambiguous state discrimination was already realized experimentally. The first experiment, designed for the discrimination of two linearly polarized states of light, was done by Huttner *et al.* [17]. There are also some newer proposals of optical implementations [18]. The interest in the quantum state discrimination is not only ''academic,'' unambiguous state discrimination can be used, e.g., as an efficient attack in quantum cryptography [19].

III. ''UNIVERSAL'' DISCRIMINATOR

Let us suppose that we want to discriminate unambiguously between two known nonorthogonal states. However, we would like to have a possibility to ''switch'' the apparatus in order to be able to work with several different pairs of states.

Let us have two (nonorthogonal) input states of a qubit. We can always choose such a basis that they read $\alpha_0|0_D\rangle \pm\beta_0|1_D\rangle$ with $\alpha_0=\cos(\varphi_0/2)$ and $\beta_0=\sin(\varphi_0/2)$; the value of φ_0 can be from 0 to $\pi/2$ (φ_0 is the angle between the two states). Let us have one additional ancillary qubit, initially in a state $|0_A\rangle$. On both the ''data'' and the ancilla we apply the following unitary transformation $\mathcal{U}_{DA}$:

$$\begin{aligned} |0_D0_A\rangle &\rightarrow \cos\theta|0_D0_A\rangle+\sin\theta|0_D1_A\rangle,\\ |1_D0_A\rangle &\rightarrow |1_D0_A\rangle,\\ |0_D1_A\rangle &\rightarrow -\sin\theta|0_D0_A\rangle+\cos\theta|0_D1_A\rangle,\\ |1_D1_A\rangle &\rightarrow |1_D1_A\rangle, \end{aligned} \tag{2}$$

where $\cos\theta=\tan(\varphi_0/2)$. If we then make a von Neumann measurement consisting of the projectors $\mathsf{P}_+=|+\rangle\langle+|$, $\mathsf{P}_- =|-\rangle\langle-|$, and $\mathsf{P}_0=\mathbb{1}-\mathsf{P}_+-\mathsf{P}_-$, where

$$|\pm\rangle=(|0_D0_A\rangle\pm|1_D0_A\rangle)/\sqrt{2}, \tag{3}$$

we can unambiguously determine the input state (with a certain probability of success) [17]. This measurement is optimal in the sense that the probability of an inconclusive result

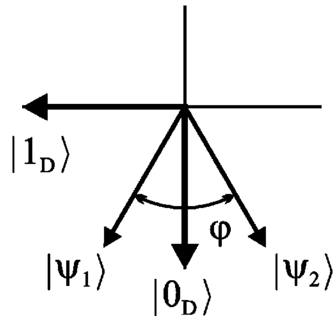

FIG. 1. The states $|\psi_1\rangle$ and $|\psi_2\rangle$ [defined by Eq. (4)] with real coefficients α and β can be visualized in a two-dimensional real space. The angle φ is related to the overlap of the two states: $\langle\phi_1|\phi_2\rangle=\cos\varphi=|\alpha|^2+|\beta|^2=2|\alpha|^2-1$.

is the lowest possible (and it is the same for both states). The probability of the successful discrimination is $2\sin^2(\varphi_0/2)$ [14].

Let us suppose now the set of pairs

$$|\psi_1\rangle=\alpha|0_D\rangle+\beta|1_D\rangle,$$
$$|\psi_2\rangle=\alpha|0_D\rangle-\beta|1_D\rangle, \tag{4}$$

where $\alpha=\cos(\varphi/2)$ and $\beta=\sin(\varphi/2)$, for all φ from the interval $(0,\pi)$. That is, we consider all pairs of states that lie on a real plane and that are located symmetrically around the state $|0_D\rangle$; see Fig. 1. Further, let us suppose that the ancillary qubit is allowed to be in an arbitrary pure state

$$|\Xi\rangle_A=a|0_A\rangle+b|1_A\rangle. \tag{5}$$

Thus the total input state reads

$$|\Psi\rangle_{DA}=(\alpha|0_D\rangle\pm\beta|1_D\rangle)\otimes(a|0_A\rangle+b|1_A\rangle)=\alpha a|0_D0_A\rangle$$
$$+\alpha b|0_D1_A\rangle\pm\beta a|1_D0_A\rangle\pm\beta b|1_D1_A\rangle. \tag{6}$$

After the action of transformation (2) on this state one obtains the resulting state in the following form [the transformation is fixed for all φ; still $\cos\theta=\tan(\varphi_0/2)$]

$$\mathcal{U}_{DA}|\Psi\rangle_{DA}=(\alpha a\cos\theta-\alpha b\sin\theta)|0_D0_A\rangle+(\alpha a\sin\theta$$
$$+\alpha b\cos\theta)|0_D1_A\rangle\pm\beta a|1_D0_A\rangle\pm\beta b|1_D1_A\rangle. \tag{7}$$

If the coefficients a and b in the state of the ancilla are chosen in such a way that

$$(\alpha a\cos\theta-\alpha b\sin\theta)=\beta a:=q/\sqrt{2} \tag{8}$$

then the expression (7) simplifies to the form

$$\mathcal{U}_{DA}|\Psi\rangle_{DA}=q|\pm\rangle+\text{const}_1|0_D1_A\rangle\pm\text{const}_2|1_D1_A\rangle, \tag{9}$$

where the states $|\pm\rangle$ are defined by Eq. (3). Clearly, applying the projective measurement introduced above one is able to discriminate unambiguously states (4) for any given $\varphi\in(0,\pi)$ provided he/she has prepared the proper state of the ancilla. The first term in Eq. (9) corresponds to the successful discrimination, while the last two terms correspond to inconclusive results. The probability of success is

$$P_{succ}=|q|^2=P_{opt}R(\varphi,\varphi_0)=2\sin^2\frac{\varphi}{2}R(\varphi,\varphi_0), \tag{10}$$

where

$$R(\varphi,\varphi_0)=\frac{\cos\varphi_0(\cos\varphi+1)}{1+\cos\varphi_0-\sin\varphi\sin\varphi_0} \tag{11}$$

is the ratio between the actual value of the probability of successful discrimination and its optimal value. This expression is obtained from the condition (8) together with the normalization relation $|a|^2+|b|^2=1$.

From above it follows that it is possible to implement a "universal quantum multimeter" that is able to discriminate probabilistically but unambiguously (with no errors) between two nonorthogonal states for the large class of nonorthogonal pairs. The selection of the desired regime (i.e., the selection of the pair of states that should be unambiguously discriminate) is done by the choice of the quantum state of the ancillary qubit. This program state selects the measurement to be performed on the system. The probability of the successful discrimination can be optimal only for one such pair of states.

In the limit case when $\varphi_0=0$, i.e., $\theta=\pi/2$ (this is the fixed parameter of the employed unitary transformation), the probability of the successful discrimination for different φ's (i.e., for different settings of the ancilla and different pairs of input states) is the same as in the "quasi-classical" case, $P_{succ}=\frac{1}{2}\sin^2\varphi$. By a quasiclassical approach we mean the probabilistic measurement when one randomly selects [20] the projective measurement in one of two orthogonal basis that both span the two-dimensional space containing both nonorthogonal states of interest (4). One basis consists of the state $|\psi_1\rangle$ and its orthogonal complement $|\psi_1^{\perp}\rangle$. If one finds the result corresponding to $|\psi_1^{\perp}\rangle$ he/she can be sure that the state $|\psi_1\rangle$ was not present. Analogously, the other basis consists of the state $|\psi_2\rangle$ and its orthogonal complement.

On the other hand when $\varphi_0=\pi/2$, i.e., $\theta=0$, there is no way to fulfill the condition (8) with $a\neq0$ (and $P_{succ}\neq0$) unless $\alpha=\beta=1/\sqrt{2}$. That is, only two orthogonal states (3) can be unambiguously discriminated.

If the parameter φ_0 is somewhere in between 0 and $\pi/2$ the probability of success (as a function of φ) is very close to the optimal value in the relatively large vicinity of φ_0; see Fig. 2. However, for small values of φ it goes below the success probability of the quasiclassical case and for $\varphi=\pi/2$ (orthogonal states) the probability of successful discrimination is lower than unity.

One can ask for the optimal value of φ_0 in the sense that the average probability of successful discrimination [or, alternatively, function $R(\varphi,\varphi_0)$] over some chosen interval of φ's is maximal. For example, if we are interested in the average value of $R(\varphi,\varphi_0)$ over the interval of φ from 0 to $\pi/2$ we find that it is maximized when $\varphi_0\approx0.235\pi$ (the corresponding average value of R is 0.92).

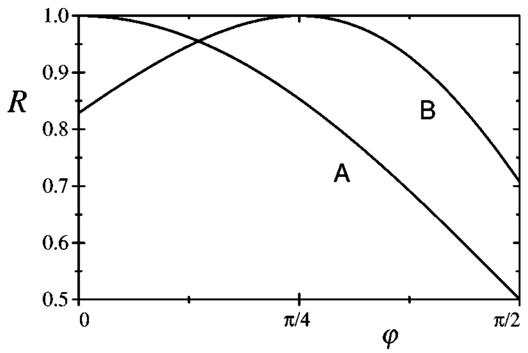

FIG. 2. The ratio $R(\varphi,\varphi_0)$ of the actual probability of successful discrimination to the optimal value of this probability as a function of the angle φ between two considered state vectors. Curve A shows the "quasiclassical" limit $(\varphi_0=0)$. Curve B represents the case when $\varphi_0=\pi/4$.

For pedagogical reasons, until now we have only worked with the states from a particular real subspace of the Hilbert space of the data qubit. However, it should be stressed that the method works for any two "input" states that are symmetrically displaced with respect to $|0_D\rangle$. In other words, the condition (8) can be fulfilled for any complex α and β. Simply,

$$\frac{b}{a}=\frac{1}{\sin\theta}\left(\cos\theta-\frac{\beta}{\alpha}\right).$$

The probability of the successful discrimination of states then reads

$$P_{\rm succ}=\frac{2\sin\theta|\alpha\beta|^2}{1-2\cos\theta\,{\rm Re}(\alpha\beta)}, \tag{12}$$

where ${\rm Re}(\alpha\beta)$ denotes the real part of $\alpha\beta$.

IV. CONCLUSIONS

We have proposed a programmable quantum measurement device for the error-free discrimination of two nonorthogonal states of qubit that works with a large set of pairs of states. The device can be set to discriminate unambiguously any two states that are symmetrically located around some fixed state [in the sense of Eq. (4)]. The setting is done through the state of a program register that is represented by another qubit. This means that the particular pair of states that can be unambiguously discriminated is specified by the state of a "program" qubit. Two possible input states of the "data qubit" that are in correspondence with the program setting are never wrongly identified but from time to time we can get an inconclusive result. The probability of successful discrimination is optimal only for one program setting. However, the device can be designed in such a way that the probability of successful discrimination is very close to the optimal value for a relatively large set of program settings. Let us stress the *quantum nature* of the "programming:" The states of the program register that represent different programs can be *nonorthogonal*.

We have also discussed some general questions concerning the possibilities to build multipurpose quantum measurement devices ("quantum multimeters") that could perform a required POVM depending on a quantum state of their program register. Most of these questions remain unanswered. For instance, let us suppose a set of all POVMs that can be obtained if we combine the measured system with an ancilla of some fixed dimension in an arbitrary state and carry out an arbitrary projective (von Neumann) measurement on the composite system. This is equivalent to carrying out an arbitrary unitary operation followed by some fixed projective measurement. Imagine now that we can change only the state of the ancilla but our projective measurement (or unitary transformation) is fixed. The question is: What measurement (operation) do we need to approximate all the POVMs from the set introduced above with the maximal average fidelity? Apparently, this question raises the other interesting task: How to define the distance between two POVMs? Such problems are not trivial, however, they open perspectives in investigation of programmable quantum devices.

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[20] With the same probabilities provided that the frequencies of the occurrence of the input states are also the same.