Reconstruction of motional states of neutral atoms via maximum entropy principle

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We present a scheme for a reconstruction of states of quantum systems from incomplete tomographiclike data. The proposed scheme is based on the Jaynes principle of *maximum entropy*. We apply our algorithm for a reconstruction of motional quantum states of neutral atoms. As an example we analyze the experimental data obtained by Salomon and co-workers and we reconstruct Wigner functions of motional quantum states of Cs atoms trapped in an optical lattice.

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I. INTRODUCTION

A reconstruction of states of quantum systems from experimental data represents an important tool for a verification of predictions of quantum theory. It also allows us to check the fidelity of quantum state preparation as well to study the fidelity of processing of information encoded in states of quantum systems. Complementary discrepancies between estimated (reconstructed) states based on the measured data and theoretical predictions can serve as an indicator of various noise sources that occur during the quantum information processing or in the measurement of quantum states. Without *a priori* assumptions about the character of physical processes and properties of reconstruction schemes the reconstruction cannot distinguish between imperfections related to an incoherent quantum state processing and nonideal measurements [1]. Determination of limits for coherent control of quantum degrees of freedom or identification of sources of decoherence are essential for systems that are considered for quantum computing and information processing [2].

In atomic optics a highly coherent control of motional degrees of freedom has been achieved for trapped ions [3] and recently also for neutral atoms [4,5]. Cold atoms can be cooled into specific quantum states within microwells of an optical lattice that is induced by laser beams. Cold neutral atoms in optical lattices represent a promising system for quantum information processing. To verify a degree (fidelity) of coherent control over motional degrees of freedom of neutral atoms a reconstruction of their motional quantum states from measured data has to be considered. We develop a reconstruction procedure based on the Jaynes principle of maximum entropy (*maxent*) [6–8] to achieve this goal. This scheme allows us to perform a state reconstruction from the experimental data obtained by Salomon and co-workers.

In Sec. II we present a brief description of the reconstruction procedure of a density operator of a quantum system based on the Jaynes principle of maximum entropy. In Sec. III we utilize the maxent principle for development of the reconstruction scheme of motional states of atoms and we perform numerical tests of our approach. Our reconstruction scheme is applied to the experimental data in Sec. IV.

II. MAXENT PRINCIPLE AND RECONSTRUCTION OF DENSITY OPERATORS

Let us assume a set of observables $\hat{G}_\nu$ $(\nu=1,\ldots,n)$ associated with the quantum system under consideration. This system is prepared in an unknown state $\hat{\rho}$. Let us assume that from a measurement performed over the system mean values $\overline{G}_\nu$ of the observables $\hat{G}_\nu$ are found. The task is to determine (estimate) the unknown state of the quantum system based on the results of the measurement. Providing the set of the observables $\hat{G}_\nu$ is not equal to the *quorum* (i.e., the complete set of system observables [9]), then the measured mean values do not determine the state uniquely. Specifically, there is a large number of density operators that fulfill the conditions

$$\mathrm{Tr}\,\hat{\rho}_{\{\hat{G}\}}=1,$$

$$\mathrm{Tr}\,(\hat{\rho}_{\{\hat{G}\}}\hat{G}_\nu)=\overline{G}_\nu,\quad \nu=1,2,\ldots,n, \tag{2.1}$$

that is, the normalization condition and the constraints imposed by the results of the measurement. To estimate the unknown density operator in the most reliable way we utilize the Jaynes principle of maximum entropy (*maxent* principle) [6–8,10] according to which among those operators that fulfill the constraints (2.1) the most reliable reconstruction (estimation) $\hat{\rho}_r$ is the one with the maximal value of the von Neumann entropy $S(\hat{\rho})=-\mathrm{Tr}(\hat{\rho}\ln\hat{\rho})$:

$$S(\hat{\rho}_r)=\max[S(\hat{\rho}_{\{\hat{G}\}};\forall\hat{\rho}_{\{\hat{G}\}}]. \tag{2.2}$$

As shown by Jaynes [6] the operator that fulfills the constraints (2.1) and simultaneously maximizes the von Neumann entropy can be expressed in the generalized canonical form

$$\hat{\rho}_r=\frac{1}{Z_{\{\hat{G}\}}}\exp\left(-\sum_\nu \lambda_\nu\hat{G}_\nu\right), \tag{2.3}$$

where

$$Z_{\{\hat{G}\}}(\lambda_1,\ldots,\lambda_n)=\mathrm{Tr}\left[\exp\left(-\sum_\nu \lambda_\nu \hat{G}_\nu\right)\right] \tag{2.4}$$

is the generalized partition function and λ_ν are the Lagrange multipliers. The Lagrange multipliers λ_ν are chosen so that the density operator (2.3) fulfills the constraints (2.1) imposed by the results of the measurement. It is then obvious that the mean values $\overline{G_\nu}$ that determine the density operator are related to the Lagrange multipliers via the derivatives of the partition function

$$\overline{G_\nu}=\mathrm{Tr}(\hat{\rho}_r\hat{G}_\nu)=-\frac{\partial}{\partial\lambda_\nu}\ln Z_{\{\hat{G}\}}(\lambda_1,\ldots,\lambda_n). \tag{2.5}$$

If we solve the last equation with respect to the Lagrange multipliers we can express them in terms of the measured mean values

$$\lambda_\nu=\lambda_\nu(\overline{G_1},\ldots,\overline{G_n}). \tag{2.6}$$

When we substitute the Lagrange multipliers (2.6) into the expression for the generalized canonical density operator (2.3) we obtain the explicit expression for the reconstructed (estimated) density operator.

As a typical example of the application of the maxent principle we can consider a measurement of a single-mode electromagnetic field, modeled as a harmonic oscillator. Imagine, that as a result of the measurement we know the mean photon number $\overline{n}$ in the given field mode. Certainly, there are (infinitely) many quantum states of a single-mode electromagnetic field (e.g., a Fock state, a coherent state, a squeezed state, etc.) with the given mean photon number. So the question is: Which is the best (most reliable) estimation of the measured state given the mean photon number is known? The mean photon number is in some sense the least available information about the measured state. Consequently, the state is least determined. On the other hand, pure states are completely determined, which is reflected by the fact that they have a zero von Neumann entropy. Therefore, we expect that the most reliable reconstruction in the given case is a statistical mixture that is determined just by a single parameter—the mean photon number. It is well known that a statistical mixture, which is parametrized just by a single parameter, is a thermal state. This statistical mixture of Fock states is characterized by a temperature, or the corresponding mean photon number. In addition, for a given temperature (mean photon number) the thermal state exhibits the largest von Neumann entropy. Consequently, from the Jaynes principle of the maximum entropy it follows that if from a measurement only a mean photon number is known that the most reliable estimation of the measured state is the thermal state.

The maxent principle is not the only criterion how to choose an appropriate density operator among those $\hat{\rho}_{\{\hat{G}\}}$ that fulfill the constraints (2.1). Based on an intuition or some addition *a priori* knowledge one can apply other criteria. For example, the *maximum likelihood* principle has been adopted successfully for estimation of quantum states [12]. Although this reconstruction scheme can result in nonphysical estimations (e.g., density operators that are not properly normalized, etc.). In general, a consistent reconstruction scheme has to avoid nonphysical results (e.g., occurrence of negative probabilities). In the maxent reconstruction a physical estimate is guaranteed by the canonical form of the density operator (2.3). The maxent principle is *the most conservative assignment in the sense that it does not permit one to draw any conclusions not warranted by the data*. From this point of view the maxent principle has a very close relation (or can be understood as the generalization) of Laplace's principle of *indifference*, which states that where nothing is known one should choose a constant valued function to reflect this ignorance. Then it is just a question how to quantify a degree of this ignorance. If we choose an entropy to quantify the ignorance, then the relation between the Laplace's indifference principle and the Jaynes principle of the maximum entropy is transparent, i.e., for a constant-valued probability distribution the entropy takes its maximum value.

The maxent reconstruction has been applied for various quantum systems, such as light field mode, spin systems [11]. In what follows we adopt it for the reconstruction of vibrational states of neutral atoms. We assume the experimental setup realized Salomon and co-workers [4,5].

III. RECONSTRUCTION OF MOTIONAL STATES OF NEUTRAL ATOMS

Recently, experimental manipulations of motional quantum states of neutral atoms have been reported by Salomon and co-workers [4,5]. Cold Cs atoms can be cooled into specific quantum states of a far detuned one-dimensional (1D) optical lattice. The optical lattice is induced by the interference of two laser beams. Along the vertical z axis a periodic potential of "harmonic" microwells is produced with a period of 665 nm and with an amplitude of about 0.2 μK [5]. The vertical oscillation frequency in a microwell at the center of the trap is $\omega_z/2\pi=85$ kHz. The corresponding ground state has the rms size $\Delta z_0=\sqrt{\hbar/2m\omega_z}\approx 21$ nm and $\Delta p_0/m=\sqrt{\hbar\omega_z/2m}\approx 11$ mm/s is its rms velocity width. The trapped cloud of neutral Cs atoms has a nearly Gaussian shape with a vertical rms size $\Delta\xi_0=53$ μm. With the help of deterministic manipulations the neutral atoms can be prepared in nonclassical 1D motional states along the vertical axis such as squeezed states, number states, or specific superpositions of number states [5]. The measurement of the prepared quantum state $\hat{\rho}$ is performed as follows: The system is evolved within the harmonic potential during the time τ. Then the lasers are turned off and the system undergoes the ballistic expansion (BE). After the time of flight $T=8.7$ ms a 2D absorption image of the cloud is taken in 50 μs with a horizontal beam [5]. Integration of 2D absorption images in the horizontal direction gives us the spatial distribution along the vertical z axis. Therefore, we will consider only 1D quantum-mechanical system along the vertical axis.

To confirm that a desired quantum state has been obtained (engineered) one can compare the spatial distributions along the vertical axis with the predicted ones. The coincidence of these spatial distributions is a necessary but not the sufficient requirement. A complete verification of the fidelity of the preparation of desired quantum states requires a quantum state reconstruction procedure. In order to perform this task we adopt the maxent principle [11]. To do so we utilize a close analogy between quantum homodyne tomography [14] and the BE absorption imaging for the case of the pointlike cloud (with the rms size equal to zero).

A. Quantum tomography via the maxent principle

Quantum tomography is based on the inverse Radon transformation of the measured probability density distributions $w_{\hat{\rho}}(x_\theta)$ for rotated quadratures $\hat{x}_\theta=(1/\sqrt{2})(\hat{a}e^{-i\theta}+\hat{a}^\dagger e^{i\theta})$ [13,14]. These distributions can be represented as a result of the measurement of the continuous set of projectors $|x_\theta\rangle\langle x_\theta|$. Based on the measurement of the distributions $w_{\hat{\rho}}(x_\theta)$ for all values of $\theta\in[0,\pi]$ we can formally reconstruct the density operator according to the formula [15]

$$\hat{\rho}_r=\frac{1}{Z_0}\exp\left[-\int_0^\pi d\theta\int_{-\infty}^{\infty}dx_\theta|x_\theta\rangle\langle x_\theta|\lambda(x_\theta)\right], \quad (3.1)$$

where the Lagrange multipliers $\lambda(x_\theta)$ are given by an infinite set of equations,

$$w_{\hat{\rho}}(x_\theta)=\sqrt{2\pi}\langle x_\theta|\hat{\rho}_r|x_\theta\rangle, \quad \forall x_\theta\in(-\infty,\infty). \quad (3.2)$$

If the distributions $w_{\hat{\rho}}(x_\theta)$ are measured for all values of x_θ and all angles θ then the density operator $\hat{\rho}_r$ is reconstructed precisely and is equal to density operator obtained with the help of the inverse Radon transformation or with the help of the pattern functions (for more details see [15]).

In practical experimental situations (e.g., see the experiments by Smithey *et al.* [16] and by Schiller *et al.* [17]) it is impossible to measure the distributions $w_{\hat{\rho}}(x_\theta)$ for all values of x_θ and all angles θ. What is measured are distributions (histograms) for finite number N_θ quadrature angles θ and the finite number N_x of "bins" for quadrature operators. This means that practical experiments are associated with an observation level specified by a *finite* number of observables

$$\hat{F}_{jk}=|x_{\theta_k}^{(j)}\rangle\langle x_{\theta_k}^{(j)}|, \quad (3.3)$$

with the number of quadrature angles equal to N_θ and the number of bins for each quadrature equal to N_x. We can, therefore, assume that from the measurement of the observables $\hat{F}_{jk}$ the mean values $\overline{F}_{jk}$ are determined (these mean values correspond to "discretized" quadrature distributions). In addition it is usually the case that the mean excitation number of the state is known (measured) as well.

The operators $\hat{F}_{jk}$ together with $\hat{n}$ form a specific observation level corresponding to the *incomplete tomographic measurement*. In this case we can express the generalized canonical density operator in the form

$$\hat{\rho}_r=\frac{1}{Z}\exp\left(-\lambda_n\hat{n}-\sum_{j=1}^{N_x}\sum_{k=1}^{N_\theta}\lambda_{j,k}\hat{F}_{jk}\right). \quad (3.4)$$

The knowledge of the mean photon number is essential for the *maxent* reconstruction because it formally regularizes the *maxent* reconstruction scheme (the generalized partition function is finite in this case).

B. Motional states of atoms via the maxent principle: Formalism

In the quantum homodyne tomography the probability distributions are measured for the rotated quadrature operators $\hat{x}_\theta$. The annihilation and creation operators of motional quanta, $\hat{a}$ and $\hat{a}^\dagger$, are related to the position and momentum operators, $\hat{z}$ and $\hat{p}$, via expressions $\hat{z}=(1/\sqrt{2})(\hat{a}+\hat{a}^\dagger)$ and $\hat{p}=(1/\sqrt{2})i(\hat{a}-\hat{a}^\dagger)$, respectively. The angle θ of the quadrature operator corresponds to $\omega_z\tau$ and vertical "cuts" of the absorption images (taken after the BE) can be associated with quadrature probability distributions. However, for a real physical situation with a nonzero rms size of the cloud the vertical "cuts" of absorption images correspond to a coarse-grained quadrature probability distributions. In particular, the vertical cuts of measured absorption images (taken in 2D) give us (after integration along the horizontal direction) the spatial distribution along the vertical axis. The spatial distribution along the vertical z axis can be expressed as

$$\overline{F}_\tau(z)=T^{-1}\int F_0(\xi_0)P_\tau[(z-\xi_0)/T]d\xi_0, \quad (3.5)$$

where $F_0(\xi_0)$ is the initial spatial distribution of the cloud in the z direction (i.e., a Gaussian distribution with the rms size $\Delta\xi_0$). The function $P_\tau(v)$ denotes the velocity probability distribution of the measured quantum state that has been evolved for time τ in the harmonic potential before the BE, i.e.,

$$P_\tau(v)=|\langle v|\psi(\tau)\rangle|^2, \quad |\psi(\tau)\rangle=\hat{U}(\tau)|\psi(0)\rangle. \quad (3.6)$$

Here $\hat{U}(\tau)=\exp(-i\hat{H}\tau/\hbar)$ represents the time-evolution operator for the harmonic oscillator with the Hamiltonian $\hat{H}=\hat{p}^2/2m+m\omega_z^2\hat{z}^2/2$. Now we can treat the measured "cuts" as the mean values of specific observables: $\overline{F}_\tau(z)=\mathrm{Tr}[\hat{\rho}\hat{F}_\tau(z)]$. In practice just a few discrete times τ_j ($j=1,\ldots,N_\tau$) are considered and the z coordinate is discretized into the bins $z_k(k=-N_z,\ldots,N_z)$ of a given resolution Δz. The set of operators that enters the Eq. (3.4) for the maxent reconstruction then takes the form

$$\hat{F}_{\tau_j}(z_k)=T^{-1}\int F_0(\xi_0)\hat{U}^\dagger(\tau_j)\left|\frac{z_k-\xi_0}{T}\right\rangle\times\left\langle\frac{z_k-\xi_0}{T}\right|\hat{U}(\tau_j)d\xi_0$$
$$(j=1,\ldots,N_\tau;k=-N_z,\ldots,N_z). \quad (3.7)$$

We have already commented that the operator of mean phonon number $\hat{n}$ is added to the set of observables $\{\hat{F}_{\tau_j}(z_k)\}$. Knowledge of the mean excitation number $\overline{n}$ is essential in the case of an incomplete set of observables [11]. Knowledge of the mean excitation number leads to a natural "truncation" of the Hilbert space. The inclusion of the mean phonon number into the maxent reconstruction scheme does not represent its limitation as the mean energy represents one of basic characteristics of any system that should be inferred from the measurement.

The experimental "cuts" of the BE absorption images $[\hat{F}_{\tau_j}(z)]$ can be taken at few selected times, for example, $\omega_z\tau_j=0,\pi/4,\pi/2,3\pi/4$ $(N_\tau=4)$. To perform the reconstruction we have to determine the Lagrange multipliers $\{\lambda_{j,k}\}$ and λ_n associated with $\{\hat{F}_{\tau_j}(z_k)\}$ and $\hat{n}$, respectively, in the expression for the generalized canonical density operator (3.4). The Lagrange multipliers can be determined via the minimization of a deviation function ΔF with respect to the measured data, i.e.,

$$\Delta F=\sum_{j,k} w_{j,k}\{\overline{F}_{\tau_j}(z_k)-\mathrm{Tr}[\hat{\rho}_r\hat{F}_{\tau_j}(z_k)]\}^2 + w_{\overline{n}}\{\overline{n}-\mathrm{Tr}(\hat{\rho}_r\hat{n})\}^2. \tag{3.8}$$

Here $\{w_{j,k}\}$ and $w_{\overline{n}}$ represent positive weight factors for particular observables. Without any prior knowledge about the state we can take for simplicity $w_{i,j}=1$. The weight factor $w_{\overline{n}}$ associated with the mean phonon number can be chosen according to our preference either to fit better the "cuts" of the BE images or the mean phonon number. In the case of the perfect measurement and the complete reconstruction the result has to be independent of the choice of the weight factors (in this case we can take $w_{\overline{n}}=1$). The weight factors could be also associated with the *prior* information about the dispersion of the measured observables. In particular, the weight factors can be taken as $w_\nu\sim\sigma_\nu^{-2}$ to reflect the knowledge of variances σ_ν for the measured observables $\hat{G}_\nu$. When the mean values of the observables for the maxent estimate $\hat{\rho}_r$ fit within desired interval $\overline{G}_\nu\pm\sigma_\nu$ then contributions of the observables to the deviation function ΔF are of the same order (~ 1). However, in our case we do not assume the knowledge of variances for the measured discretized probability distributions (taking $w_\nu=1$).

Once the Lagrange multipliers are numerically fitted, the result of the reconstruction—the generalized canonical density operator $\hat{\rho}_r$—can be visualized, for example, via the corresponding Wigner function [18] that can be defined as a particular Fourier transform of the density operator $\hat{\rho}$ of a harmonic oscillator expressed in the basis of the eigenvectors $|q\rangle$ of the position operator $\hat{q}$,

$$W_{\hat{\rho}}(q,p)\equiv\int_{-\infty}^{\infty}d\zeta\langle q-\zeta/2|\hat{\rho}|q+\zeta/2\rangle e^{ip\zeta}. \tag{3.9}$$

C. Numerical simulation

To test our reconstruction procedure let us consider the reconstruction of the Wigner function of the motional quantum state $|\psi(0)\rangle=(|0\rangle+|1\rangle)/\sqrt{2}$ of Cs atoms trapped in the optical lattice. This kind of state has been demonstrated in recent experiments [5]. We assume the following setup parameters: $\omega_z/2\pi=80$ kHz, the rms size of the ground state $\Delta z_0=22$ nm, the rms velocity width $\Delta p_0/m=11$ mm/s and the rms width of the cloud of the atoms about 60 μm. Before BE (with BE time $T=8.7$ ms) the atoms evolve within the harmonic trapping potential for $\tau=0,1.6,3.2,4.8$ μs. As the input for the reconstruction via the maxent principle four vertical "ideal" cuts of the BE absorption images are taken as shown in Fig. 1(b). In addition, for the phonon number operator $\hat{n}$ that is included in the set of measured observables (see discussion above) we assume the mean value $\overline{n}=0.5$. The result of the ideal reconstruction is shown in Fig. 1. The fidelity of the measured and the reconstructed quantum states is close to unity that means a perfect reconstruction with $\Delta F=10^{-10}$, entropy $S=10^{-7}$, $\Delta\rho=10^{-8}$ has been achieved. Here $\Delta\rho=\Sigma_{m,n}|(\hat{\rho}-\hat{\rho}_r)_{mn}|^2$ denotes a deviation of the original and reconstructed density operators.

Obviously, in a real measurement the measured values are always fluctuating around the exact ones due to an experimental noise. Therefore, we simulate a nonideal measurement introducing random fluctuations to the measured values of observables. It means that instead of the ideal values $\overline{F}_{\tau_j}(z_k)$ we use for the maxent reconstruction procedure the fluctuating ("noisy") values

$$\overline{F}'_{\tau_j}(z_k)=\overline{F}_{\tau_j}(z_k)+\eta\xi_{j,k}[\overline{F}_{\tau_j}(z_k)]^{1/2}. \tag{3.10}$$

Here η is a relative-error parameter that characterizes the quality of the measurement and $\{\xi_{j,k}\}$ represents a Gaussian noise for observables. The result of the reconstruction is shown in Fig. 2 for $\eta=0.1$. Noisy mean values of the observables are shown in Fig. 2(b). Despite a significant relative error the reconstruction is almost perfect with the fidelity of the measured and the reconstructed states still close to one $(\Delta F=0.16$, entropy $S=0.01$, $\Delta\rho=0.05)$. The minimum value of the deviation function $\Delta F=0.16$ can serve also as a measure of the imperfection of the given measurement (due to a technical noise) [20].

A typical nonclassical state that we can utilize for a further test is the even coherent state $\mathcal{N}_e(|\alpha\rangle+|-\alpha\rangle)$ that is a superposition of two coherent states with opposite phases [19]. For the amplitude $\alpha=\sqrt{2}$ we obtained $\Delta F=10^{-8}$, the entropy $S=0.026$ and $\Delta\rho=10^{-4}$ [under assumption that the exact mean phonon number $\overline{n}=1.928$ is known (Fig. 3)]. In the case of the imperfect measurement with $\eta=0.1$ the reconstruction leads to $\Delta F=0.14$, entropy $S=0.13$, and $\Delta\rho=0.06$ for $\overline{n}=2.09$. The fidelity of the reconstructed and the measured states is in this case also close to one (Fig. 4).

In order to model a technical noise in the measurement we have been considered Gaussian fluctuations proportional to the square root of the mean values. It means that tails of the "cuts" of BE images do not introduce a significant error [compare Fig. 1(b) and Fig. 2(b)]. However, in the current measurements the situation seems to be different and the fluctuations do not decrease with the amplitude of the expected values.

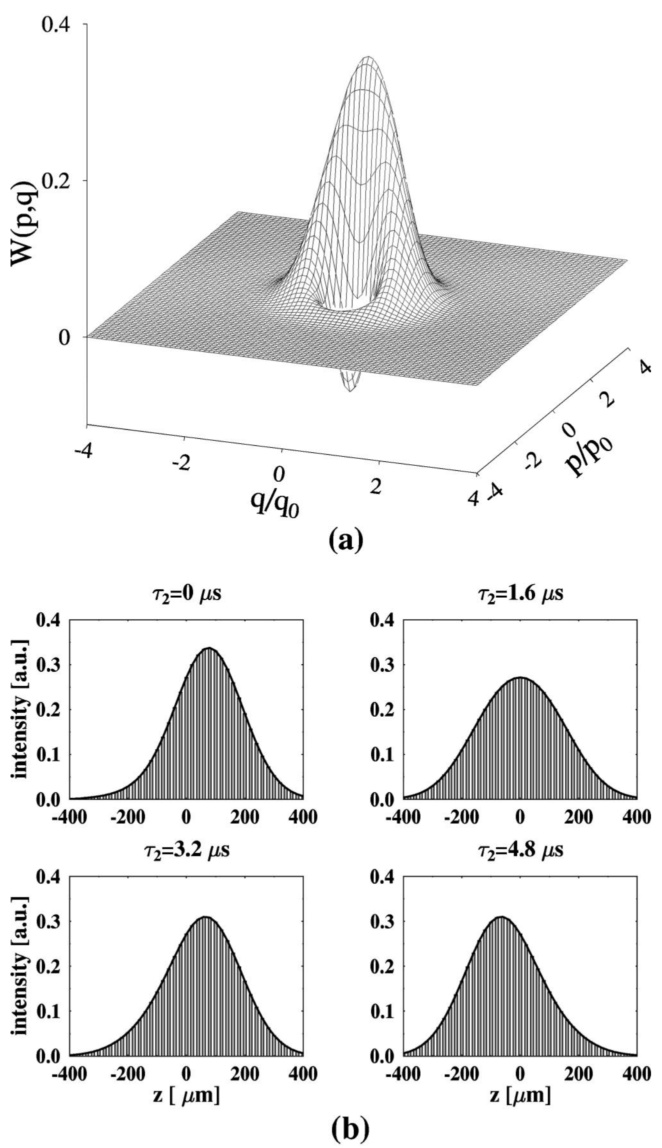

FIG. 1. (a) Numerical simulation of the reconstruction of the Wigner function of the motional quantum state $(|0\rangle+|1\rangle)/\sqrt{2}$ of Cs atoms trapped in the optical lattice (assuming $\omega_z/2\pi=80$ kHz, the rms size of the ground state $\Delta z_0=22$ nm, and the rms velocity width $\Delta p_0/m=11$ mm/s). For the reconstruction via the maxent principle four vertical cuts of the absorption images (with BE time $T=8.7$ ms) have been taken (b). The histograms correspond to the measured data while the solid lines are obtained from the reconstructed Wigner function (i.e., they correspond to reconstructed marginal distributions). Before BE the atoms evolve within the trapping potential for the times $\tau=0,1.6,3.2,4.8$ μs. In addition, the mean number of motional quanta $\overline{n}=0.5$ and the rms width of the cloud of the atoms about 60 μm have been assumed.

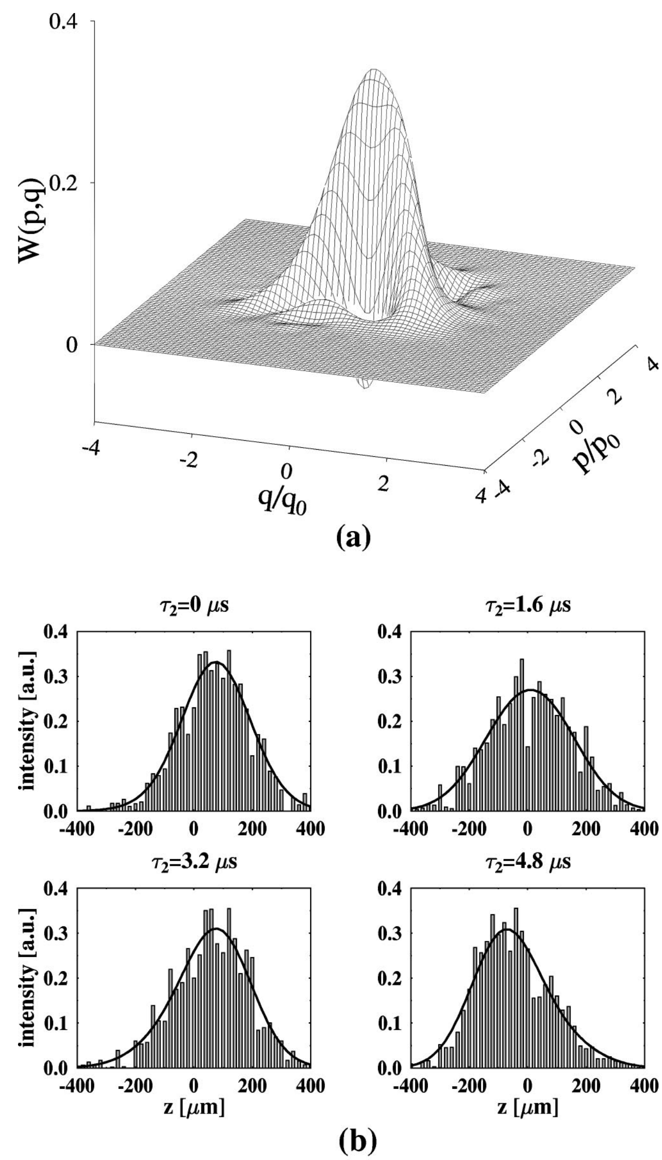

FIG. 2. (a) Numerical simulation of the reconstruction of the Wigner function of the atomic motional quantum state $(|0\rangle+|1\rangle)/\sqrt{2}$ for the same settings as in Fig. 1. (b) Four vertical cuts of the absorption images taken for reconstruction are fluctuating randomly around their ideal values shown in Fig. 1(b) with the relative error $\eta=0.1$. The histograms correspond to the measured data while the solid lines are obtained from the reconstructed Wigner function. In addition, the mean phonon number $\overline{n'}=0.6$ has been considered.

The fundamental question in the context of the *maxent* reconstruction of states from incomplete tomographic data is whether the quality of the reconstruction can be improved using additional data from subsequent time moments τ and how many such time moments τ are required for the complete reconstruction of the unknown state $\hat{\rho}$. We have shown recently [15] that for the quantum tomography just *three* quadrature distribution are sufficient for a complete reconstruction using the maxent principle (in the case of the perfect measurement). This corresponds to the ideal case without the spatial dispersion of the cloud of atoms, i.e., the choice with $\omega_z\tau_j=0,\pi/4,\pi/2$ $(N_\tau=3)$ is sufficient for $\Delta\xi_0\to 0$. Obviously, in experiments with neutral atoms the spatial size of the atomic cloud is nonzero. However, in the case of the ideal measurement three BE absorption images associated with three "rotations" $\omega_z\tau_j$ are still sufficient for a complete reconstruction of tested examples of quantum states. On the other hand, it seems that for higher mean phonon numbers the spatial distributions along the vertical axis

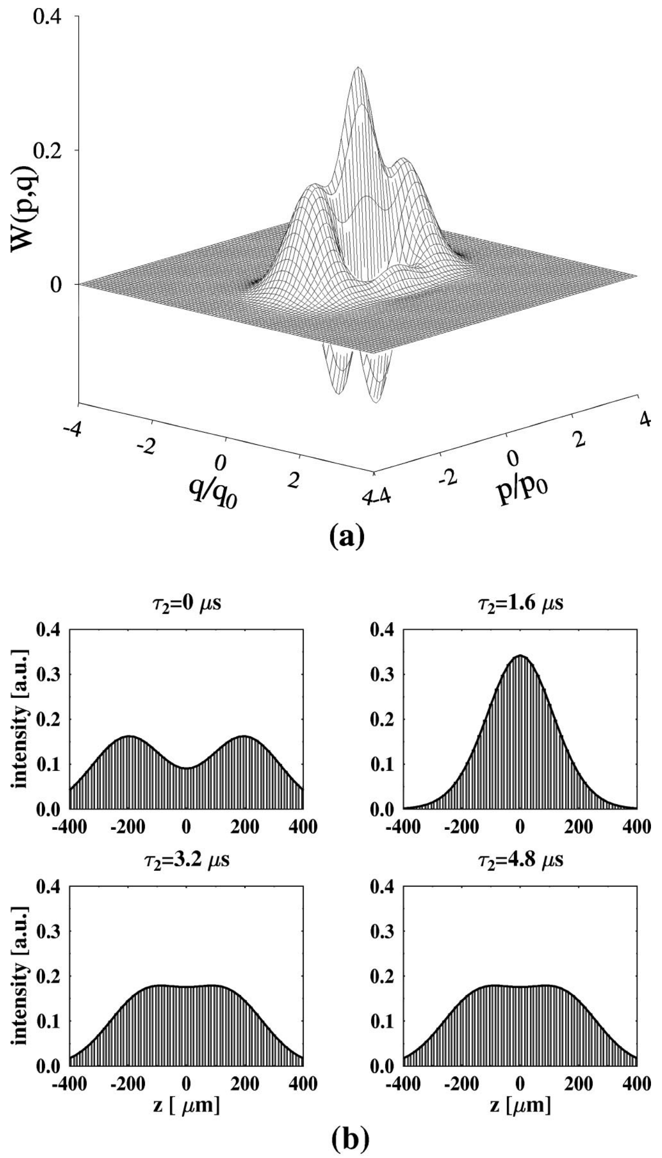

FIG. 3. Numerical simulation of the reconstruction of the Wigner function of the motional quantum state $\mathcal{N}_e(|\alpha\rangle+|-\alpha\rangle)$ with $\alpha=\sqrt{2}$ for the case of the ideal measurement. The mean number of motional quanta $\overline{n}=1.928$. Other settings are the same as in Fig. 1.

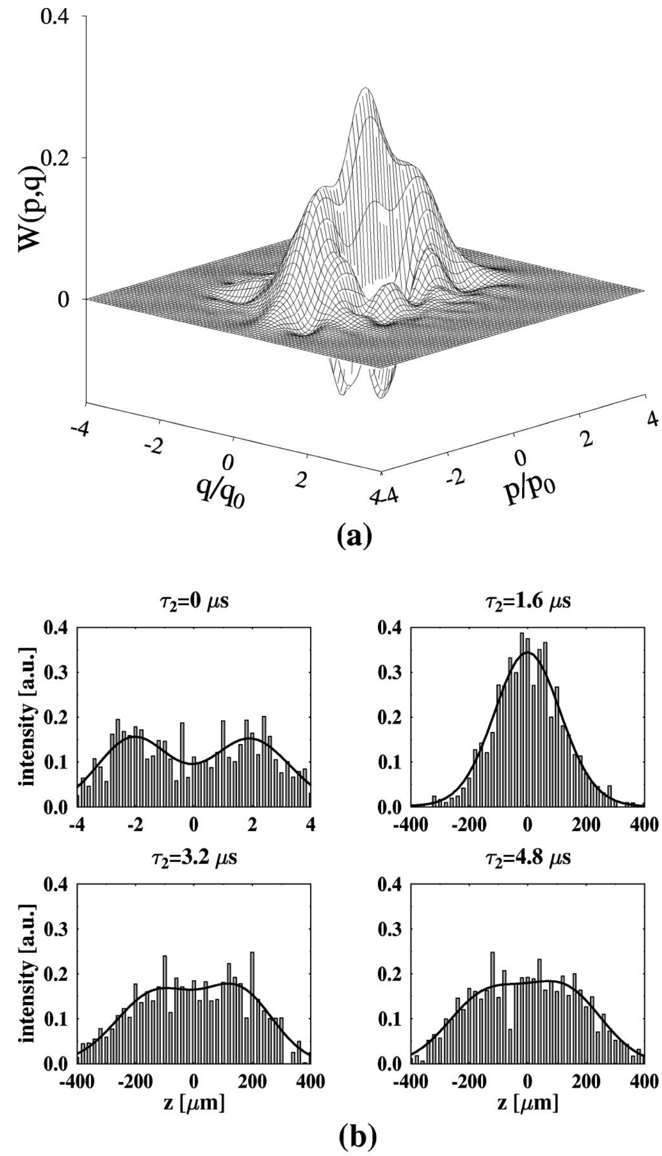

FIG. 4. Numerical simulation of the reconstruction of the Wigner function of the motional quantum state $\mathcal{N}_e(|\alpha\rangle+|-\alpha\rangle)$ with $\alpha=\sqrt{2}$ for the case of noisy measurement with $\eta=0.1$. The "measured" mean number of the motional quanta $\overline{n'}=2.09$. Other settings are the same as in Fig. 2.

that are directly determined from absorption images should be known with improving precision (and on a wider interval of values as well). In the above examples we have considered for convenience BE images for four "rotations" ($N_\tau=4$) that results in a very good reconstruction.

IV. RECONSTRUCTION FROM EXPERIMENTAL DATA

In what follows we will apply the maxent reconstruction scheme to the data obtained by Salomon and Bouchoule [21]. First we note that the unknown quantum state should belong to a Hilbert subspace that can be determined easily. Thus we can limit ourselves to the subspace spanned by Fock (number) states $|0\rangle,|1\rangle,\ldots,|N-1\rangle$. The upper bound on the accessible phonon number N is given by experimental limitations such as, for example, a feasible depth of microwells of the optical lattice and the validity of the harmonic potential approximation. For recent experiments N has been typically of the order of 10. This value is large enough to demonstrate the preparation of many nonclassical states but on the other hand excludes highly squeezed states from a coherent processing.

Let us consider the experimental arrangement used by Salomon and Bouchoule [21] with the parameters: $\omega_z/2\pi=80$ kHz, the rms size of the ground state $\Delta z_0=22$ nm, the rms velocity width $\Delta p_0/m=11$ mm/s, the rms width of the cloud of the atoms about 60 μm and BE time $T=8.7$ ms. Initially the atoms are prepared in a well-defined motional state $|\psi_0\rangle$ (e.g., in the vacuum state $|0\rangle$). Then the optical lattice is switched off for the time period t_1 during which the atoms evolve freely towards the state $|\psi_1\rangle = \exp(-it_1\hat{p}^2/2m)|\psi_0\rangle$. Next, the optical lattice is again switched on for the time τ during which the atoms evolve within the harmonic trapping potential. The measurement is performed after the BE time. The first two stages can be considered as the preparation of the state $|\psi_1\rangle$. After its "rotation" by $\omega_z\tau$ (within the phase space of the harmonic oscillator) and the subsequent BE the absorption images are taken.

The considered data are for the initial vacuum state, which means that under ideal conditions a squeezed state $|\psi_1\rangle = \exp(-it_1\hat{p}^2/2m)|0\rangle$ should be prepared. The vertical spatial distributions obtained from the measured 2D-absorption images are discretized into pixels (bins) with the pixel width 5.45 μm. The optical density of each pixel is averaged in the horizontal direction in which the absorption images are divided into 50 rows, each 3.9 μm wide (these rows cover the size of the cloud in the horizontal direction). For the reconstruction via the maxent principle four vertical spatial distributions for "rotation" times $\tau = 0$, 1.6, 3.2, and 4.8 μs are taken. The selected times roughly correspond to rotations within the phase space by $\omega_z\tau = 0, \pi/4, \pi/2$ and $3\pi/4$, respectively. Unfortunately, the mean excitation number $\hat{n}$ for measured state $|\psi_1\rangle$ was not measured explicitly in the experiment, therefore, we have to estimate it as follows: During the free expansion period the rms size of the cloud increases by $\Delta x = p_0\tau_1/m$. The corresponding increase of the potential energy $\frac{1}{2}m\omega_z^2(\Delta x)^2$ in units $\hbar\omega_z$ gives us the increase of the number of excitation quanta with respect to the initial state $|\psi_0\rangle$. For $\tau_1 = 4$ μ and the initial vacuum it means $\overline{n} \approx 1$. Experiments can be realized also for higher τ_1. For example, $\tau_1 = 8$ μ leads to $\overline{n} \approx 4$. However, as mentioned above, such "squeezed" states with a significant contribution of higher phonon number states violate the underlying harmonic approximation for the potential. To keep a coherent control an anharmonic part of the potential has to be taken into account.

The result of the reconstruction via the maxent principle is shown in Fig. 5. The deviation of the fitted and measured values is $\Delta F = 0.09$ and the entropy of the reconstructed state $S = 1.0$. It means that the reconstructed state is a statistical mixture. We see a two peak structure, which suggests that there is a mixture of two squeezed states coherently displaced from each other. It is caused by the fact that the vertical center of the cloud was not fixed in the experiment and it has to be determined by our fit for each measured BE absorption image separately. Assuming *a priori* knowledge that the Wigner function has a symmetric shape with respect to the origin of the phase space (i.e., there is no coherent amplitude) a Gaussian fit can be used to determine the center of the cloud for each vertical distribution. For states with a nonzero coherent amplitude the center of the cloud should be fixed already in the experiment.

It turns out that the reconstruction results do not describe the squeezed vacuum state as was originally expected [21]. The main reason is that the mean phonon number was not measured directly in the experiment. It can be inferred only indirectly from the ideal case without any incoherence during preparation or measurement. As we discussed above, it is essential to include the information about the mean number of vibrational quanta into the maxent reconstruction scheme. In optical tomography the analogous information about mean photon number can be obtained from distributions of two "orthogonal" quadratures. In our case it could correspond to two absorption images such that $\omega_z(\tau_j - \tau_k) = \pi/2$. However, it would require a precise timing of the evolution within the harmonic trapping potential. Therefore the mean number of

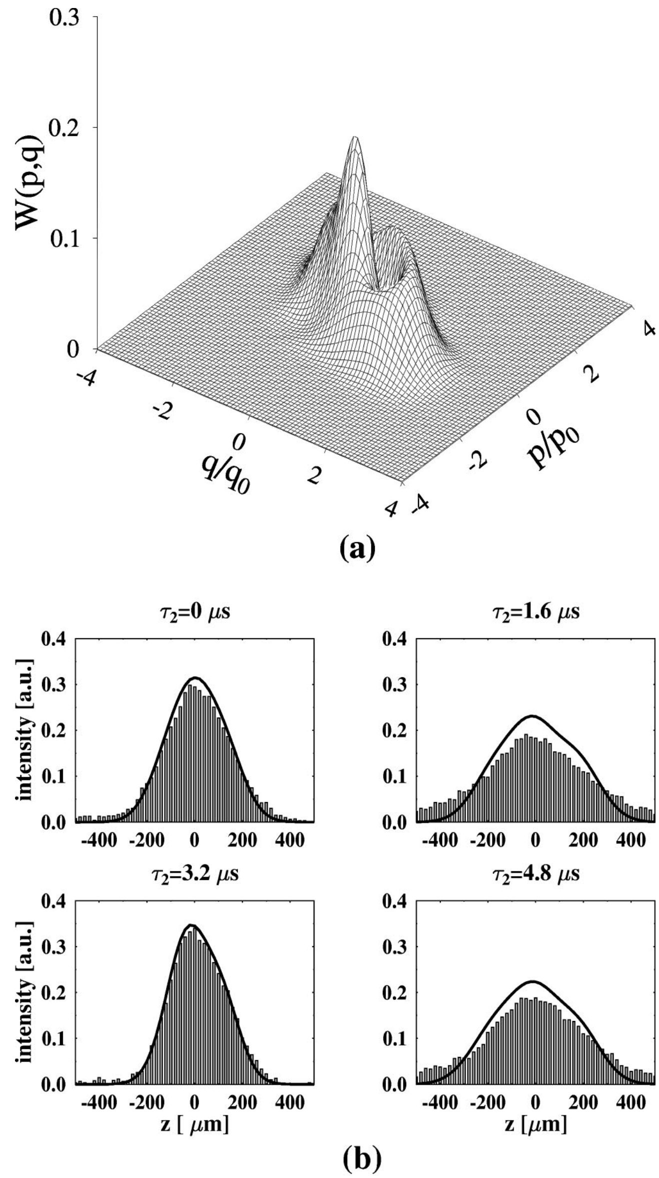

FIG. 5. The Wigner function reconstructed from the experimental data obtained by Salomon and co-workers. The experimental setting is the same as for Fig. 1. From the experimental data we have inferred the mean number of motional quanta $\overline{n} \approx 1.0$, while the reconstructed value is $\overline{n'} \approx 1.1$. Deviation of the measured and predicted values of observables is $\Delta F = 0.09$ and entropy of the reconstructed mixture state is $S = 1.0$. Subtraction of a background from the measured marginals gives almost the same Wigner function and reduces significantly a difference between measured and reconstructed marginals.

vibrational quanta should be determined in an independent measurement.

Another problem arises from a slow convergence of antisqueezed spatial distributions that are derived directly from the measured absorption images. In particular, the convergence of tails is too slow for those ''rotations'' that correspond to antisqueezed phases, i.e., $\tau = 1.6,\ 4.8\ \ \mu$s [see Fig. 5(b)]. The slow convergence is reflected by the presence of non-negligible backgrounds for Gaussian fits to these spatial distributions. If we eliminate (subtract) these backgrounds from the measured distributions the maxent reconstruction gives almost the same Wigner function as in Fig. 5(a) but with a highly reduced deviation function $\Delta F = 0.02$ (comparing to $\Delta F = 0.09$ in Fig. 5). Such background in these absorption images can be caused by an incoherence associated with a violation of the harmonic approximation. In fact, in our analysis we have neglected the change of the oscillation frequency along the z axis. In recent experiments, the oscillation frequency decreases 10% from ω_z for microwells at the edge of the initial cloud.

V. CONCLUSIONS

We have presented a very efficient reconstruction scheme for the reconstruction of motional states of atoms based on the *maxent* principle. The main advantage of the scheme is that it always results in reconstructions that are physical states (unlike in the case of the maximum likelihood estimation that can result in nonphysical estimations). Moreover, the scheme is very efficient in a sense that it requires just a small number of tomographic phase-space cuts.

We have applied the scheme for a reconstruction of motional quantum states of neutral atoms. As an example we have analyzed the experimental data obtained by Salomon and co-workers and we reconstruct the Wigner function of motional quantum states of Cs atoms trapped in the optical lattice. In our analysis we have neglected the change of the oscillation frequency along z axis in recent experiments. The dispersion of the oscillation frequency is of the order of a few percent. This source of errors can significantly affect the quality of a quantum state preparation and its reconstruction. In addition, only up to the first ten bound states of microwells of the optical lattice can be approximated by a harmonic potential. It implies limits on coherent manipulations of quantum states. It means that states with a significant contribution of higher number (Fock) states cannot be prepared and manipulated in a controlled way.

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