Wigner-function description of quantum teleportation in arbitrary dimensions and a continuous limit

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We present a unified approach to quantum teleportation in arbitrary dimensions based on the Wigner-function formalism. This approach provides us with a clear picture of all manipulations performed in the teleportation protocol. In addition within the framework of the Wigner-function formalism all the imperfections of the manipulations can be easily taken into account.

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All quantum mechanical phenomena may be described in terms of quasiprobability distributions as an alternative to the direct application of density matrices. Wigner functions are especially frequently applied, as they behave similarly to classical probability distributions from several points of view. For quantum states with infinite-dimensional Hilbert spaces, the application of Wigner functions has become a standard part of considerations. For finite-dimensional Hilbert spaces, the Wigner-function formalism was first investigated by Wootters [1]. The discrete Wigner functions have shown to be useful in investigating coherent states in a finite-dimensional basis [2], in definition of Q functions and other propensities [3], and also have played a role in the development of number-phase Wigner functions [4]. Quantum tomography for finite-dimensional Wigner functions has also been developed, applying a generalized definition [5].

A great deal of attention has been paid recently to the phenomenon of quantum teleportation, which is the basic primitive of quantum communication, and it is also interesting from the point of view of quantum nonlocality [6]. The experimental feasibility of the phenomenon [7–10] highly contributes to the importance of these investigations. The idea of quantum teleportation by Bennett *et al.* [11] was formulated on finite-dimensional Hilbert spaces. In this context, the conventional description applying Hilbert-space vectors is appropriate. On the other hand, the idea of continuous-variable quantum teleportation, proposed originally by Vaidman [12], was first put into a quantum optical context by Braunstein and Kimble using the Wigner-function formalism [13]. However, this scheme may also be described in terms of either wave functions [14,15] or Fock states [16], and a low-dimensional coherent state description has also been developed recently [17]. A covariant description in terms of canonically conjugate observables and their eigenstates is also possible [18], providing a description valid for both discrete and continuous dimensions.

In this paper we present the description of quantum teleportation purely in the framework of the Wigner-function formalism of quantum mechanics. The main emphasis is put on the case of finite-dimensional Hilbert spaces, but we make some comments on the infinite-dimensional limits. It will be shown that the entire process of quantum teleportation can be consistently described purely in terms of Wigner functions, and in this context, the finite- and infinite-dimensional cases can be treated in a conceptually uniform way.

The paper is organized as follows. After a brief review of some elements of finite-dimensional Wigner-function formalism, we describe the ideal Einstein-Podolsky-Rosen state. Then the entire teleportation process is discussed, and conclusions are drawn.

Consider a physical system with states described by the N-dimensional Hilbert space $\mathcal{H}$. We define two noncommuting Hermitian operators $\hat{q}$ and $\hat{p}$ describing two canonically conjugate quantities. We will call them "position" and "momentum," respectively, though they may be realized by several physical quantities, as, for instance, photon number and Pegg-Barnett phase operators on a truncated Fock space. The operators are defined as

$$\hat{q}=\sum_{k=0}^{N-1} k|k\rangle\langle k|, \quad \hat{p}=\sum_{l=0}^{N-1} l|p_l\rangle\langle p_l|, \tag{1}$$

where the set of $|k\rangle$ position and $|p_l\rangle$ momentum eigenstates both form an orthonormal basis on $\mathcal{H}$, and

$$|p_l\rangle=\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} e^{i(2\pi/N)kl}|k\rangle \tag{2}$$

holds.

Wigner functions for this discrete system can be defined in a slightly different manner depending on the properties of the number N, the dimensionality of the corresponding Hilbert space. In what follows we will suppose that N is greater than 3 and it is a prime number. Though it introduces some loss of generality, apart from technical details, there is no significant physical difference between the cases discussed and the remaining two possibilities. In case of $N=2$, a different definition of the Wigner function has to be applied, while for composite N's, the phase spaces are Cartesian products of lower-dimensional phase spaces. Alternatively, one may use the formalism suggested in Ref. [5].

According to the original paper of Wootters [1], the Wigner function corresponding to a state in a Hilbert space with dimension $N \geqslant 3$ prime is defined with the aid of the discrete Wigner operator

$$\hat{A}(q,p)=\sum_{r,s} \delta_{2k,r+s} \exp\left[i \frac{2\pi}{N} p(r-s)\right] |r\rangle\langle s|, \tag{3}$$

where q and p take integer values from 0 to $N-1$. The (q,p) pairs constitute the discrete phase space. For a state described by a density matrix ϱ the Wigner function is

$$W(q,p)=\frac{1}{N}\mathrm{Tr}(\varrho \hat{A}). \tag{4}$$

Wigner functions defined in this way obey analogous properties to those defined on infinite-dimensional Hilbert spaces. The marginal distributions of the functions

$$P_q(q)=\sum_p W(q,p), \quad P_p(p)=\sum_q W(q,p) \tag{5}$$

describe the statistics of measurements of observables $\hat{q}$ and $\hat{p}$, respectively.

For multipartite systems, Wigner functions are defined, similarly to the infinite-dimensional case, with the expectation values of the direct product of the Wigner operators. In what follows we consider multipartite systems with Hilbert spaces of equal dimension. For a bipartite system with subsystems 1 and 2, described by the joint density matrix $\varrho^{(12)}$,

$$W(q_1,p_1,q_2,p_2)=\frac{1}{N^2}\mathrm{Tr}[\varrho^{(12)}\hat{A}_1(q_1,p_1)\otimes \hat{A}_2(q_2,p_2)] \tag{6}$$

Wigner functions describing a subsystem are obtained by summing the joint Wigner function in the corresponding set of the respective variables, e.g., from Eq. (6) we have

$$\begin{aligned} W(q_1,p_1)&=\sum_{q_2,p_2=0}^{N-1} W(q_1,p_1,q_2,p_2),\\ W(q_2,p_2)&=\sum_{q_1,p_1=0}^{N-1} W(q_1,p_1,q_2,p_2). \end{aligned} \tag{7}$$

For bipartite systems, the completely entangled Bell states

$$|\Xi_{p,x}\rangle_{12}=\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} e^{i(2\pi/N)kp}|k\rangle_1|k-x\rangle_2 \tag{8}$$

form an orthonormal basis on the $\mathcal{H}\otimes\mathcal{H}$ Hilbert space of the joint system. These are common eigenstates of the following joint observables:

$$(\hat{q}_1-\hat{q}_2)|\Xi_{p,x}\rangle_{12}=(q_1-q_2)|\Xi_{p,x}\rangle_{12}, \tag{9}$$

$$(\hat{p}_1+\hat{p}_2)|\Xi_{p,x}\rangle_{12}=(p_1+p_2)|\Xi_{p,x}\rangle_{12}.$$

Following Bennett [11], we shall suppose that the sender Alice and the receiver Bob share the subsystems 2 and 3 in the entangled state

$$|\Xi_{0,0}\rangle_{23}=\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} |k\rangle_2|k\rangle_3. \tag{10}$$

In what follows, we shall use the term "EPR state" for this state. The Wigner function of this state can be calculated according to Eqs. (3), (4), and (6) and is found to be

$$W_{\mathrm{EPR}}(q_2,p_2,q_3,p_3)=\frac{1}{N^2}\delta_{q_2,q_3}\delta_{p_2,-p_3}. \tag{11}$$

Calculating the Wigner functions for subsystems 2 and 3 according to Eq. (7), both of them are found to be the constant $1/N^2$. From this it follows that any of the marginals describe a uniform distribution. This reflects the EPR nature of the state: making observations on either of the subsystems separately, both position and momentum have random values. On the other hand, according to Eq. (9), some joint observables have a definite value as it is also clearly reflected by Eq. (11): $q_2-q_3=0$ and $p_2+p_3=0$. From this we may conclude that the form of the EPR Wigner function in Eq. (11) could have been even a plausible ansatz.

The Wigner function in Eq. (11) shows the connection with the EPR state used by Braunstein and Kimble for continuous-variable teleportation. In the continuous-variable case for an ideal EPR state, Dirac deltas appear corresponding to a state with infinite energy. Therefore instead of the ideal EPR state, usually two-mode squeezed vacuum is considered instead, which results in the imperfection of the protocol.

Let us consider the teleportation process. Alice, the sender, and Bob, the receiver, have shared the EPR pair described by the Wigner function in Eq. (11). In addition Alice has system 1 in the arbitrary state described by a Wigner function $W_{\mathrm{in}}(q_1,p_1)$. The joint Wigner function of the whole system is thus

$$W(q_1,p_1,q_2,p_2,q_3,p_3)=\frac{1}{N^2}W_{\mathrm{in}}(q_1,p_1)\delta_{q_2,q_3}\delta_{p_2,-p_3}. \tag{12}$$

Alice has to carry out a projective measurement on subsystems 1 and 2. This measurement is performed in the Bell basis, which obviously projects the systems 1 and 2 on the Bell states (8). As we have already mentioned, these states are simultaneous eigenstates of the joint observables $\hat{X}_2=\hat{q}_1-\hat{q}_2$ and $\hat{P}_1=\hat{p}_1+\hat{p}_2$. In order to describe the measurement, we have to express the Wigner function in Eq. (12) in terms of these variables and $\hat{X}_1=\hat{q}_1+\hat{q}_2$ and $\hat{P}_2=\hat{p}_1-\hat{p}_2$, instead of q_1,p_1 and q_2,p_2. Note, that because of the modulo-N arithmetics, the ranges of the new variables are the same.

This canonical transformation is more straightforward in the infinite-dimensional case, where we can introduce a $\sqrt{2}/2$ factor in the definition of the new variables, and thus it is easy to express the inverse transformation in the same fashion. In our case, a division by 2 appears in the inverse formula, which seems to be inappropriate at first sight. This problem can be overcome in the following way: As N is odd, we may introduce a "generalized division by 2" in the modulo-N sense as

$$\mathcal{D}_2(k)=\begin{cases}\dfrac{k}{2}, & n \ \text{even}\\[2mm] \dfrac{k+N}{2}, & n \ \text{odd},\end{cases} \tag{13}$$

which has the property $2\mathcal{D}_2(k)=k$. Here we emphasize again that *all* additions, subtractions, and multiplications are understood in the modulo-N sense. With the aid of this operation, the old variables can be expressed as

$$\begin{aligned} q_1&=\mathcal{D}_2(X_1+X_2), \quad q_2=\mathcal{D}_2(X_1-X_2)\\ p_1&=\mathcal{D}_2(P_1+P_2), \quad p_2=\mathcal{D}_2(P_1-P_2).\end{aligned} \tag{14}$$

The Wigner function in Eq. (12) after the transformation is

$$\begin{aligned} W&(X_1,P_1,X_2,P_2,q_3,p_3)\\ &=\frac{1}{N^2}\delta_{X_1-X_2,2q_3}\delta_{P_1-P_2,-2p_3}\\ &\quad\times W_{\text{in}}(\mathcal{D}_2(X_1+X_2),\mathcal{D}_2(P_1+P_2)).\end{aligned} \tag{15}$$

At this stage, all subsystems are entangled. Note, that the canonical transformation, which is described here by introducing new variables, is physically a unitary transformation that entangles two subsystems and it even cannot be carried out completely by using linear optical elements [19].

Now we are ready to describe the Bell-state measurement, which results in values X_2 and P_1, the classical information that is sent to Bob. Summing the Wigner function in Eq. (15) in variables X_1,P_2,p_3,q_3, we obtain the probability distribution of the measurement results, which is equal to constant $1/N^2$. Thus we can obtain each possible measurement result with equal probability, in accordance with Bennett's description.

To describe the conditional projection by the measurement, we have to keep variables X_2 and P_1 constants, as these numbers constitute the result of the measurement, and we have to sum the Wigner function of Eq. (15) in variables X_1 and P_2, as we lose all information about these because of the projective measurement. This procedure is the exact analog of the continuous case. The resulting Wigner function has to be renormalized and it has the form

$$W_{\text{out}}(q_3,p_3)=W_{\text{in}}(q_3+X_2,p_3+P_1). \tag{16}$$

It is seen, that the resulting Wigner function is a shifted version of the original, and the shift is determined by the result of the measurement. This is the exact analog of the continuous case. Bob, possessing the values X_2 and P_1, can restore the teleported state. The shift in a finite-dimensional Hilbert space is illustrated in Fig. 1. Obviously, these shifts correspond to translations (canonical transformations) in a discrete phase space.

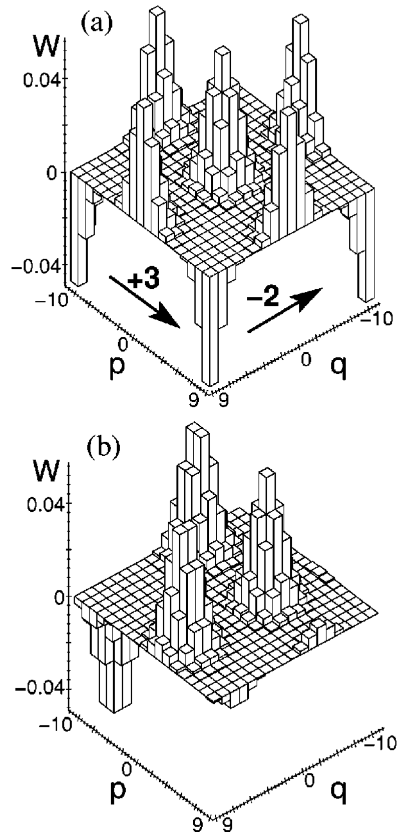

FIG. 1. Shifting of Wigner function in a discrete phase space of a quantum system with a 19-dimensional Hilbert space. (a) shows the state, which is a discrete counterpart of the harmonic-oscillator ground state (see Ref. [3]). (b) is the shifted version according to the arrows in (a). Points of the phase space are indexed so that the main peak is centered in the origin of a phase space; recall the modulo-N summation. We use units such that both q and p are dimensionless.

The required inverse transformation as described by Bennett is

$$U_{X_2,P_1}=\sum_k e^{i(2\pi/N)P_1k}|k\rangle\langle k-X_2|. \tag{17}$$

It is easy to verify that this transformation acts on a Wigner function as

$$W'(q,p)=\langle U^\dagger_{X_2,P_1}A(q,p)U_{X_2,P_1}\rangle=W(q-X_2,p-P_1), \tag{18}$$

thus our description is perfectly consistent with Bennett's results.

The similarity of our discussion to the original description of continuous-variable quantum teleportation by Braunstein and Kimble is apparent. Care should be taken, however, if the actual infinite-dimensional limit is to be constructed from the description above, which is far from straightforward indeed. For instance, several nontrivial problems have to be overcome if $\hat{q}$ and $\hat{p}$ is associated with photon numbers and the Pegg-Barnett phase [20,21].

In conclusion, we have shown that quantum teleportation can be described purely in terms of Wigner functions, and this could have been possible even without mentioning the underlying Hilbert space. This approach has several advantages in the description of imperfections. Noisy entanglement can be treated, similarly to the continuous case, by replacing the Kronecker deltas describing ideal entangled states with the appropriate Wigner function. While projective measurement is described by filtering with delta functions here, a fuzzy measurement may be described by filtering with unsharp filters. This example suggests that Wigner functions may prove to be a useful tool for investigating phenomena in multipartite systems with finite-dimensional Hilbert spaces.

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