

## Quantum disentanglers

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It is not possible to disentangle a qubit in an *unknown* state  $|\psi\rangle$  from a set of  $N-1$  ancilla qubits prepared in a specific reference state  $|0\rangle$ . That is, it is not possible to *perfectly* perform the transformation  $(|\psi, 0, \dots, 0\rangle + |0, \psi, \dots, 0\rangle + \dots + |0, 0, \dots, \psi\rangle) \rightarrow |0, \dots, 0\rangle \otimes |\psi\rangle$ . The question is then how well we can do? We consider a number of different methods of extracting an unknown state from an entangled state formed from that qubit and a set of ancilla qubits in a known state. Measuring the whole system is, as expected, the least effective method. We present various quantum “devices” which disentangle the unknown qubit from the set of ancilla qubits. In particular, we present the *optimal universal* disentangler which disentangles the unknown qubit with a fidelity that does not depend on the state of the qubit, and a probabilistic disentangler which performs the perfect disentangling transformation, but with a probability less than 1.

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### I. INTRODUCTION

Information encoded in qubits can be used for reliable quantum communication or efficient quantum computing [1,2]. This information is encoded in a quantum state  $|\psi(\vartheta, \varphi)\rangle$  which in the case of a qubit can be parametrized as

$$|\psi(\vartheta, \varphi)\rangle = \cos\frac{\vartheta}{2}|0\rangle + e^{i\varphi}\sin\frac{\vartheta}{2}|1\rangle, \quad (1.1)$$

where  $|0\rangle$  and  $|1\rangle$  are basis vectors of the two-dimensional space of the qubit and  $0 \leq \vartheta \leq \pi$ ;  $0 \leq \varphi \leq 2\pi$ .

Qubits are very fragile, that is the state of a qubit can easily be changed by the influence of the environment or a random error. One (very inefficient) way to protect the quantum information encoded in a qubit is to measure it. With the help of an optimal measurement one can estimate the state of a qubit, with an average fidelity equal to  $2/3$  (see below). In this way a quantum information is transformed into a classical information which can be stored, copied, and processed according the laws of classical physics with arbitrarily high precision. However, in order to utilize the full potential of quantum information processing we have to keep the information in states of quantum systems, but then we are forced to face the problem of decoherence. Recently it has been proposed that quantum information and quantum information processing can be stabilized via symmetrization [3]. In particular, the qubit in an unknown state is entangled with a set of  $N-1$  (ancilla) qubits in a specific reference state (let us say  $|0\rangle$ ) so the symmetric state  $|\Psi\rangle$  of  $N$  qubits

$$|\Psi\rangle \simeq (|\psi, 0, \dots, 0\rangle + |0, \psi, \dots, 0\rangle + \dots + |0, 0, \dots, \psi\rangle), \quad (1.2)$$

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is generated. If we introduce a notation for completely symmetric states  $|N; l\rangle$  of  $N$  qubits with  $l$  of them being in the state  $|1\rangle$  and  $N-l$  of them in the state  $|0\rangle$ , then the state (1.2) can be expressed in the simple form

$$|\Psi(\bar{\vartheta}, \bar{\varphi})\rangle = \cos\frac{\bar{\vartheta}}{2}|N; 0\rangle + e^{i\bar{\varphi}}\sin\frac{\bar{\vartheta}}{2}|N; 1\rangle, \quad (1.3)$$

where the parameters  $\bar{\vartheta}$  and  $\bar{\varphi}$  are specified by the relations

$$\cos\frac{\bar{\vartheta}}{2} = \frac{\sqrt{N}\cos(\vartheta/2)}{\sqrt{\sin^2(\vartheta/2) + N\cos^2(\vartheta/2)}} \quad (1.4)$$

and  $\sin\bar{\vartheta}/2 = \sqrt{1 - \cos^2(\bar{\vartheta}/2)}$ , while  $\bar{\varphi} = \varphi$ . We see that symmetric  $N$  qubit state  $|\Psi(\bar{\vartheta}, \bar{\varphi})\rangle$  is isomorphic to a single qubit state, but in this case the information is spread among  $N$  entangled qubits—the original quantum information is “diluted.” Each of the qubits of the  $N$ -qubit state (1.3) is in the state

$$\rho_j = \frac{N-1}{N}|0\rangle\langle 0| + \frac{(1-\sqrt{N})}{N}\left(\cos^2\frac{\bar{\vartheta}}{2}|0\rangle\langle 0| + \sin^2\frac{\bar{\vartheta}}{2}|1\rangle\langle 1|\right) + \frac{1}{\sqrt{N}}|\psi(\bar{\vartheta}, \bar{\varphi})\rangle\langle\psi(\bar{\vartheta}, \bar{\varphi})|.$$

We define the average fidelity between the single state  $\rho_j$  and the original qubit  $|\psi(\vartheta; \varphi)\rangle$  as

$$\bar{\mathcal{F}} = \int d\Omega \langle\psi(\vartheta; \varphi)|\rho_j(\bar{\vartheta}, \bar{\varphi})|\psi(\vartheta; \varphi)\rangle, \quad (1.5)$$

where  $d\Omega = \sin\vartheta d\vartheta d\varphi/4\pi$  is the invariant measure on the state space of the original qubit [i.e., we assume no *prior* knowledge about the pure state  $|\psi(\vartheta; \varphi)\rangle$ ]. For this fidelity we find the expression

$$\bar{\mathcal{F}}_0 = \frac{N^2 - 1 - 2\ln N}{2(N-1)^2}. \quad (1.6)$$

We see that for  $N=1$  the fidelity  $\bar{\mathcal{F}}_0$  is equal to unity (as it should be, because in this case  $|\Psi\rangle=|\psi\rangle$ ) while in the limit  $N\rightarrow\infty$  we find  $\bar{\mathcal{F}}=1/2$ . In fact in this limit density operators of individual qubits are approximately equal to  $|0\rangle\langle 0|$ . In other words, individually the qubits of the symmetric state  $|\Psi(\bar{\vartheta}, \bar{\varphi})\rangle$  in the large  $N$  limit do not carry any information about the original single-qubit state  $|\psi\rangle$ . So how can we extract the information from the  $N$ -qubit symmetric state (1.3)? The ideal possibility would be to have a perfect *universal* disentangler which would perform a unitary transformation

$$|\Psi(\bar{\vartheta}, \bar{\varphi})\rangle \rightarrow |\Psi_{\text{ideal}}\rangle \equiv |N-1; 0\rangle \otimes |\psi(\vartheta, \varphi)\rangle. \quad (1.7)$$

But quantum mechanics does not allow this type of disentangling transformation [4–7].

While the perfect transformation is impossible, there are a number of things we can do to concentrate the information from the  $N$ -qubit state  $|\Psi(\bar{\vartheta}, \bar{\varphi})\rangle$  back into a single qubit. In principle, we have the following possibilities: (i) We can either optimally measure the  $N$  qubit state and based on the information obtained prepare a single-qubit state. (ii) We can design a quantum disentangler which would perform a transformation as close as possible to the ideal disentangling (1.7). In this quantum scenario we have several options—the process of disentanglement can be input-state dependent. This means that states (1.3) for some values of the parameters  $\bar{\vartheta}$  and  $\bar{\varphi}$  will be disentangled better than for other values of these parameters. Alternatively, we can construct a quantum device which disentangles all states with the same fidelity. (iii) Finally, we propose a probabilistic disentangler, such that when a specific projective measurement over an ancilla is performed at the output, the desired single-qubit state is generated. The probability of the outcome of the measurement in this case is state dependent. In what follows we shall investigate all these possibilities.

Before proceeding we note that a different type of disentangler has been considered by Terno and Mor [5–7]. They considered two different operations. The first would take the state of a bipartite quantum system and transform it into a state that is just the product of the reduced density matrixes of the two subsystems. The second, which is a generalization of the first, would again start with a state of a bipartite quantum system, and map it into a separable state which has the same reduced density matrixes as the original state. They showed that while both of these processes are impossible in general, they can be realized for particular sets of input states. Approximate disentangles have been considered by Bandyopadhyay *et al.* [10]. These authors have proposed a disentangler which maps a density operator of a bipartite quantum system into a separable state which has the reduced density matrixes *approximately* equal to the original state. The disentangles we are considering extract, to some degree of approximation, an unknown state from an entangled state formed from that state and a known state.

## II. MEASUREMENT SCENARIO

Here we first describe a measurement scenario utilizing a set of specific projection operators. Then we present the optimal measurement-based approach to quantum disentanglement and we derive an upper bound on the fidelity of the measurement-based disentangler.

We utilize the fact that the  $N$  qubit system prepared in the state  $|\Psi(\bar{\vartheta}, \bar{\varphi})\rangle$  is isomorphic to a single qubit. Therefore we first consider a strategy based on a projective measurement with two projectors [8,9]  $P_j(\vartheta', \varphi') = |\Xi_j(\vartheta', \varphi')\rangle\langle \Xi_j(\vartheta', \varphi')|$  ( $j=0,1$ ) with

$$\begin{aligned} |\Xi_0(\vartheta', \varphi')\rangle &= \cos\frac{\vartheta'}{2}|N; 0\rangle + e^{i\varphi'}\sin\frac{\vartheta'}{2}|N; 1\rangle, \\ |\Xi_1(\vartheta', \varphi')\rangle &= e^{-i\varphi'}\sin\frac{\vartheta'}{2}|N; 0\rangle - \cos\frac{\vartheta'}{2}|N; 1\rangle, \end{aligned} \quad (2.1)$$

such that  $\langle \Xi_j(\vartheta', \varphi') | \Xi_k(\vartheta', \varphi') \rangle = \delta_{j,k}$  and  $\sum_j P_j(\vartheta', \varphi') = 1$ , where the angles  $\vartheta'$  and  $\varphi'$  are chosen randomly if no *prior* information about the measured  $N$ -qubit state is available.

We can use the result of the measurement to manufacture a single-qubit state. Specifically, if the result of the measurement is positive for  $P_0$  then the single qubit is prepared in the state

$$|\eta_0(\vartheta', \varphi')\rangle = \cos\frac{\vartheta'}{2}|0\rangle + e^{i\varphi'}\sin\frac{\vartheta'}{2}|1\rangle, \quad (2.2)$$

while if the output is positive for  $P_1$  then the single qubit is prepared in the orthogonal state  $|\eta_1(\vartheta', \varphi')\rangle$ . For a particular orientation of the measurement apparatus (i.e., the angles  $\vartheta', \varphi'$ ) this measurement-based scenario gives us a single qubit prepared in the state described by the density operator

$$\rho^{(\text{meas})}(\bar{\vartheta}, \bar{\varphi}; \vartheta', \varphi') = \sum_{j=0}^1 |\langle \Psi | \Xi_j \rangle|^2 \cdot |\eta_j\rangle\langle \eta_j|. \quad (2.3)$$

After we average over all possible orientations of the measurement apparatus we obtain on average a single qubit prepared in the state

$$\rho^{(\text{est})}(\bar{\vartheta}, \bar{\varphi}) = \frac{1}{3} |\psi(\bar{\vartheta}, \bar{\varphi})\rangle\langle \psi(\bar{\vartheta}, \bar{\varphi})| + \frac{2}{3} 1. \quad (2.4)$$

To find the average fidelity of this measurement-based disentangling procedure we have to evaluate the mean fidelity  $\bar{\mathcal{F}}_1$ , that is the overlap between the state (2.4) and the original input state  $|\psi(\vartheta, \varphi)\rangle$  averaged over all possible orientations of the input qubit:

$$\bar{\mathcal{F}}_1 = \int d\Omega \langle \psi(\vartheta, \varphi) | \rho^{(\text{est})}(\bar{\vartheta}, \bar{\varphi}) | \psi(\vartheta, \varphi) \rangle. \quad (2.5)$$

Taking into account the relation (1.4) we perform the integration in Eq. (2.5) and we find

$$\bar{\mathcal{F}}_1 = \frac{1}{3} (1 + f_N), \quad (2.6)$$

where the function  $f_N$  reads

$$f_N = \frac{N^2 + 4N^{3/2} - 4N^{1/2} - 1 + 2N \ln N}{2(N-1)(N^{1/2} + 1)^2}. \quad (2.7)$$

For  $N=1$ ,  $\bar{\mathcal{F}}_1 = 2/3$  which is the optimal fidelity of estimation of the state of a single qubit. From Fig. 1 we see that the fidelity (2.6) is a decreasing function of  $N$  and in the limit  $N \rightarrow \infty$  we find  $\bar{\mathcal{F}}_1 = 1/2$ , which is equal to the fidelity of a *random* guess associated with a binary system such as the two projectors under consideration. In other words, when the original qubit is diluted into an infinite qubit state of the form (1.3) no relevant information can be gained from the measurement. The estimated density operator (2.4) in this case is simply equal to  $1/2$ , which is understandable, because as we have shown earlier in this limit the  $N$ -qubit state is approximately in the state  $|N,0\rangle$ , so information about the original is ‘‘almost’’ totally lost.

*Optimal measurement scenario.* We now want to find an upper bound  $\bar{\mathcal{F}}^{\max}$  for the average fidelity which can be achieved by a wide class of measurement-based disentanglement procedures. We assume that it is *a priori* known that our  $N$  qubit is prepared in the symmetric state (1.2) with unknown parameters  $\vartheta$  and  $\varphi$  associated with a single-qubit state (1.1). The integration measure on the state space of the single qubit is  $d\Omega = (1/4\pi) \sin \vartheta d\vartheta d\varphi$  and the corresponding prior probability density distribution on this state space is constant.

Our strategy is to measure the input state  $|\Psi\rangle$  along the vector  $|\Xi_0\rangle$  [see Eq. (2.1)], where the angles  $\vartheta'$  and  $\varphi'$  are chosen according to the distribution  $q(\vartheta', \varphi')$ , which will be left unspecified for the moment. If the answer is positive, we produce the output density matrix  $\rho_0(\vartheta', \varphi')$ , and if it is negative we produce  $\rho_1(\vartheta', \varphi')$ , where

$$\begin{aligned} \rho_j(\vartheta', \varphi') &= \int d\Omega'' p_j(\vartheta'', \varphi'' | \vartheta', \varphi') |\eta(\vartheta'', \varphi'')\rangle \langle \eta(\vartheta'', \varphi'')| \\ &= \int d\Omega'' p_j(\vartheta'', \varphi'' | \vartheta', \varphi') |\eta(\vartheta'', \varphi'')\rangle \langle \eta(\vartheta'', \varphi'')| \end{aligned} \quad (2.8)$$

with  $j=0,1$  and  $|\eta\rangle$  given by Eq. (2.2). We shall also leave the conditional probabilities,  $p_j$  unspecified, as this allows us to consider a wide range of strategies. For a fixed  $|\Xi_0\rangle$ , the probability of the output being  $\rho_0(\vartheta', \varphi')$  is  $|\langle \Xi_0 | \Psi \rangle|^2$  and the probability of it being  $\rho_1(\vartheta', \varphi')$  is  $|\langle \Xi_1 | \Psi \rangle|^2$ . Averaging over all vectors,  $|\Xi\rangle$  gives us

$$\begin{aligned} \rho^{(\text{out})}(\bar{\vartheta}, \bar{\varphi}) &= \int d\Omega' [|\langle \Xi_0 | \Psi \rangle|^2 \rho_0(\vartheta', \varphi') \\ &\quad + |\langle \Xi_1 | \Psi \rangle|^2 \rho_1(\vartheta', \varphi')] q(\vartheta', \varphi'). \end{aligned} \quad (2.9)$$

In order to find the average fidelity of the output produced by this procedure, we compute the fidelity for a particular input state and average over the input ensemble

$$\bar{\mathcal{F}} = \int d\Omega \langle \psi(\vartheta, \varphi) | \rho^{(\text{out})}(\bar{\vartheta}, \bar{\varphi}) | \psi(\vartheta, \varphi) \rangle, \quad (2.10)$$

where  $\bar{\vartheta}$  is a function of  $\vartheta$  [see Eq. (1.4)]. This can be expressed as

$$\bar{\mathcal{F}} = \int d\Omega' \int d\Omega'' \sum_{j=0}^1 P_j(\vartheta'', \varphi''; \vartheta', \varphi') f_j(\vartheta'', \varphi''; \vartheta', \varphi'), \quad (2.11)$$

where

$$P_j(\vartheta'', \varphi''; \vartheta', \varphi') = p_j(\vartheta'', \varphi'' | \vartheta', \varphi') q(\vartheta', \varphi'), \quad (2.12)$$

is a normalized joint probability distribution, and

$$\begin{aligned} f_0 &= \int d\Omega |\langle \Psi | \Xi_0 \rangle|^2 |\langle \psi | \eta \rangle|^2, \\ f_1 &= \int d\Omega |\langle \Psi | \Xi_1 \rangle|^2 |\langle \psi | \eta \rangle|^2. \end{aligned} \quad (2.13)$$

We first note that

$$\int d\Omega'' p_j(\vartheta'', \varphi''; \vartheta', \varphi') f_j(\vartheta'', \varphi''; \vartheta', \varphi') \leq h_j(\vartheta', \varphi'), \quad (2.14)$$

where

$$h_j(\vartheta', \varphi') = \sup f_j(\vartheta'', \varphi''; \vartheta', \varphi'), \quad (2.15)$$

and the supremum is taken over the variables  $\vartheta'', \varphi''$ . We then have that

$$\bar{\mathcal{F}} \leq \bar{\mathcal{F}}^{\max} = \sup [h_0(\vartheta', \varphi') + h_1(\vartheta', \varphi')], \quad (2.16)$$

where the supremum is now taken over  $0 \leq \vartheta' \leq \pi$  and  $0 \leq \varphi' < 2\pi$ .

In order to calculate this upper bound we must find explicit expressions for  $f_0$  and  $f_1$ . After performing the necessary calculations we find for  $\bar{\mathcal{F}}^{\max}$  the expression

$$\bar{\mathcal{F}}^{\max} = \frac{1}{2} \left[ 1 + \frac{\sqrt{N}}{(N-1)^3} (N^2 - 1 - 2N \ln N) \right]. \quad (2.17)$$

This fidelity for  $N=1$  is equal to  $2/3$  while in the limit  $N \rightarrow \infty$  is equal to  $1/2$ . For any other  $N$ , it is larger than the fidelity  $\bar{\mathcal{F}}_1$  of the measurement given by Eq. (2.6) as discussed in our previous example. Nevertheless, as we will show later it is always smaller than the fidelity of the universal quantum device.

### III. QUANTUM SCENARIO

In what follows we show that a quantum disentangler which preserves quantum coherences can distill the information back to a single qubit more efficiently than can the measurement-based method. As we have already said in the

introduction quantum mechanics does not allow one to construct a perfect disentangler which would perform transformation (1.7) for an arbitrary (unknown) state  $|\psi(\vartheta, \varphi)\rangle$  diluted in the  $N$  qubit symmetric state (1.3). Nevertheless, we can try to design optimal disentanglers which perform best under given constraints.

### A. State-independent devices

So let us assume our quantum disentangler  $D$  is a quantum system with a  $K$ -dimensional Hilbert space spanned by basis vectors  $|d_k\rangle$  ( $k=1, \dots, K$ ). The disentangler is always initially prepared in the state  $|d_0\rangle$ , and then it interacts with the  $N$ -qubit system in the state (1.3). At the output we want to disentangle the  $N-1$  ancilla qubits from the original qubit, so we expect to have

$$|\Psi(\bar{\vartheta}, \bar{\varphi})\rangle|d_0\rangle \rightarrow |N-1;0\rangle \otimes \sum_{k=1}^K \sum_{j=0}^1 c_j(\bar{\vartheta}, \bar{\varphi})|j\rangle|d_k\rangle. \quad (3.1)$$

As seen from Eq. (3.1) during the disentanglement process the entanglement between the  $N-1$  ancilla qubits and the original qubit is transferred (swapped) into the entanglement between the original qubit and the disentangler itself. By tracing over the disentangler we then expect to obtain the best possible disentangled qubit in the state  $\rho^{(\text{out})}(\bar{\vartheta}, \bar{\varphi})$ . Now we impose several constraints which would specify what we mean by the optimal covariant (universal) disentangler.

(1) The fidelity between the output of the disentangler and the original state  $|\psi(\vartheta, \varphi)\rangle$  has to be invariant with respect to rotations of the original qubit, so the fidelity has to be input-state independent. This universality of the disentangler would then guarantee that the information from the symmetric state (1.3) is extracted for all states equally well.

(2) We are looking for the *optimal* disentangler which would disentangle the information with the highest fidelity.

Imposing these two conditions we have found the unitary transformation which realizes the *optimal covariant* disentangler, i.e., which disentangle the qubit state  $|\psi\rangle$  from the  $N$ -qubit state  $|\Psi\rangle$  in the optimal and the  $|\psi\rangle$ -state independent way (see the Appendix). This disentangler is described by the transformation

$$\begin{aligned} |N;0\rangle|d_0\rangle &\rightarrow |N-1;0\rangle \otimes [\gamma_N|0\rangle|d_1\rangle + \delta_N|1\rangle|d_2\rangle], \\ |N;1\rangle|d_0\rangle &\rightarrow |N-1;0\rangle \otimes [\delta_N|0\rangle|d_3\rangle + \gamma_N|1\rangle|d_1\rangle], \end{aligned} \quad (3.2)$$

where  $|d_i\rangle$  are three orthonormal basis vectors of the disentangler. The amplitudes  $\gamma_N$  and  $\delta_N$  given by the relation

$$\gamma_N = \left( \frac{N+1}{2(N+1-\sqrt{N})} \right)^{1/2}, \quad \delta_N = \sqrt{1-\gamma_N^2}. \quad (3.3)$$

We can directly verify, that the fidelity  $\mathcal{F}_2 = \langle \psi(\vartheta, \varphi) | \rho_d^{(\text{out})}(\bar{\vartheta}, \bar{\varphi}) | \psi(\vartheta, \varphi) \rangle$  is input-state independent and equal to  $\mathcal{F}_2 = \gamma_N^2$ . Moreover, it can be shown that the

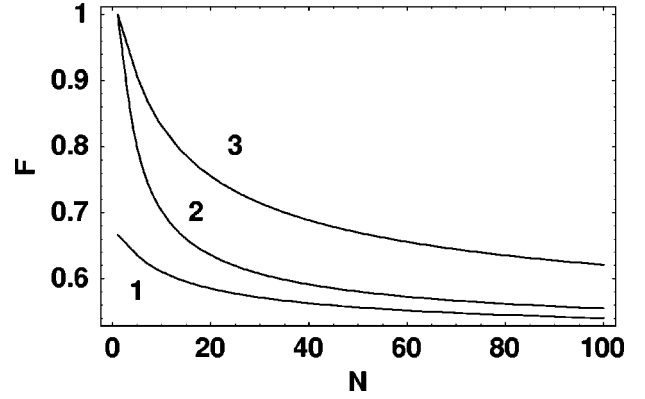


FIG. 1. Fidelities of various disentanglers as described in the text. Line 1 describes the fidelity  $\bar{\mathcal{F}}_1$  of the measurement-based disentangler given by Eq. (2.6), line 2 is for the fidelity  $\mathcal{F}_2 = \gamma_N^2$  of the universal optimal disentangler given by Eq. (3.3), and, finally line 3 is for the mean fidelity of the state-dependent disentangler via swapping  $\bar{\mathcal{F}}_3 = f_N$  given by Eq. (2.7).

transformation (3.3) is optimal, i.e., among all unitary transformations satisfying the given conditions the transformation (3.3) has the largest fidelity. We see that for  $N=1$  the fidelity  $\mathcal{F}_2 = 1$ , which is obvious, because the original qubit has not been entangled with ancilla qubits. We plot  $\mathcal{F}_2$  in Fig. 1. We see, that it is *always* larger than the fidelity of the disentanglement via measurement. In the limit  $N \rightarrow \infty$  even the quantum disentangler gives us a totally random outcome. So in this limit, even optimal quantum entangler on which we impose the universality condition, is not able to extract information from the state (1.3).

This is one of the main results of our paper—the optimal covariant quantum disentangler operates better than if the information is extracted (disentangled, distilled) from the symmetrized state (1.3) with the help of the optimal measurement. This is due to the fact that  $\bar{\mathcal{F}}^{\max} \leq \mathcal{F}_2$ .

One can also ask the opposite question, how can we generate out of a qubit in an unknown state  $|\psi\rangle$  the symmetric state of the form (1.3). It can be shown that within quantum mechanics perfect universal entanglers, which would realize the inverse of the relation (1.7) do not exist. If one wants to create a state (1.3) from a qubit in an unknown state and  $N-1$  ancilla qubits in the known state  $|0\rangle$  again two scenarios are possible, the measurement-based and quantum scenarios. It is not surprising that the quantum scenario works better. We have found the optimal universal (covariant with respect to rotations of the input qubit) quantum entangler given by the transformations

$$\begin{aligned} |0\rangle|N-1;0\rangle|e_0\rangle &\rightarrow [\gamma_N|N;0\rangle|e_1\rangle + \delta_N|N;1\rangle|e_2\rangle], \\ |1\rangle|N-1;0\rangle|e_0\rangle &\rightarrow [\delta_N|N;0\rangle|e_3\rangle + \gamma_N|N;1\rangle|e_1\rangle], \end{aligned} \quad (3.4)$$

where  $|e_k\rangle$  are three orthonormal basis states of the quantum entangler,  $|e_0\rangle$  is its initial state and the parameters  $\gamma_n$  and  $\delta_N$  are given by Eq. (3.3). One can check that the fidelity between the output of this entangler described by the density operator  $\rho_e^{(\text{out})}(\vartheta, \varphi)$  and the ideally entangled state (1.3) is

input-state independent (i.e., does not depend on the parameters  $\vartheta, \varphi$ ) and is equal to  $\gamma_N^2$ . This is the best possible universal (covariant) entangler.

### B. State-dependent devices

The universal disentangler gives a higher fidelity than does the best measurement-based procedure, but it is not obvious that this is the best that one can do. In the case of quantum cloning, the universal cloners are the ones which maximize the average fidelity [11,12]. As we shall see, however, in the case of disentanglers this is no longer the case; there are state-dependent devices which are better.

Consider the general disentangler transformation

$$\begin{aligned} |N;0\rangle|d_0\rangle &\rightarrow |N-1;0\rangle(|0\rangle|D_1\rangle + |1\rangle|D_2\rangle), \\ |N;1\rangle|d_0\rangle &\rightarrow |N-1;0\rangle(|0\rangle|D_3\rangle + |1\rangle|D_4\rangle), \end{aligned} \quad (3.5)$$

where the vectors  $|D_j\rangle$ , are states of the disentangler itself and need not be orthogonal. They must, however, satisfy the constraints imposed by the unitarity of the above transformation. The input state for the device is assumed to be  $|\Psi(\bar{\vartheta}, \bar{\varphi})\rangle$ , and the ideal output state, to which the actual output should be compared, is  $|\Psi_{\text{ideal}}\rangle = |N-1;0\rangle|\psi(\vartheta, \varphi)\rangle$ . The output state is calculated by starting with the input state, using the above transformation, and then tracing over the disentangler to obtain an output density matrix  $\rho^{(\text{out})}$ . One then finds the average fidelity for this process, which we shall call  $\bar{\mathcal{F}}_3$ , from

$$\bar{\mathcal{F}}_3 = \int d\Omega \langle \Psi_{\text{ideal}} | \rho^{(\text{out})} | \Psi_{\text{ideal}} \rangle. \quad (3.6)$$

Note that we are assuming a specific ensemble of input states; the probability of the one-qubit state  $|\psi(\vartheta, \varphi)\rangle$  is assumed to be constant on the Bloch sphere. Our result for the average fidelity for a state-dependent device depends on our choice of input ensemble, while for a state-independent device the average fidelity is independent of this ensemble.

The calculation of the average fidelity is given in the Appendix. We find that  $\|D_2\|^2 = \|D_3\|^2 = 0$  and  $|D_1\rangle = |D_4\rangle$ . This implies that the final state is just a product of the state of the  $N$  particles and the entangler state, which means that the entangler states can be dropped from the problem. Therefore, the transformation which maximizes the average fidelity is just

$$\begin{aligned} |N;0\rangle &\rightarrow |N-1;0\rangle|0\rangle, \\ |N;1\rangle &\rightarrow |N-1;0\rangle|1\rangle, \end{aligned} \quad (3.7)$$

and we have that

$$|\Psi(\bar{\vartheta}, \bar{\varphi})\rangle \otimes |0\rangle \rightarrow |N,0\rangle \otimes |\psi(\bar{\vartheta}, \bar{\varphi})\rangle, \quad (3.8)$$

which is a kind of state swapping transformation. The average fidelity itself is given by  $\bar{\mathcal{F}}_3 = f_N$ , where the coefficient  $f_N$  is given by Eq. (2.7). This average fidelity is larger than the fidelity of the optimal *universal* disentangler (see Fig. 1).

In this case, the fact that the universality condition forces us to use an additional quantum device, the disentangler, with which the qubit at the output becomes partially entangled, results in a net loss of information. As a result the fidelity of the universal (covariant) entangler is smaller.

Analogously, we find that quantum state-dependent entanglement can also be performed by a kind of state swapping transformation, i.e.,

$$|\psi(\vartheta, \varphi)\rangle \otimes |N;0\rangle \rightarrow |0\rangle \otimes |\Psi(\vartheta, \varphi)\rangle, \quad (3.9)$$

with input-state dependent fidelity  $|\langle \Psi(\vartheta, \varphi) | \Psi(\bar{\vartheta}, \bar{\varphi}) \rangle|^2$ . Nevertheless, when averaged over all values of  $\vartheta, \varphi$  we find the mean fidelity of this state-dependent entangler to be equal to  $f_N$  which on average is larger than the fidelity of the state-independent entangler.

### IV. PROBABILISTIC DISENTANGLER

Let us examine a simple quantum network which takes as an input the  $N$ -qubit state (1.3). The network is composed of a sequence of  $N-1$  controlled-NOT (C-NOT) gates  $\mathcal{P}_N = \prod_{k=1}^{N-1} C_{kN}$ , where  $C_{kl}$  is the C-NOT gate with  $k$  being the control bit and  $l$  being the target bit. This sequence of the C-NOT gates acts on the two vectors  $|N;0\rangle$  and  $|N;1\rangle$  as

$$\mathcal{P}_N |N;0\rangle \rightarrow |N-1;0\rangle|0\rangle, \quad (4.1)$$

$$\mathcal{P}_N |N;1\rangle \rightarrow \frac{1}{\sqrt{N}} (\sqrt{N-1}|N-1;1\rangle + |N-1;0\rangle)|1\rangle$$

from which it follows that the input vector (1.3) is transformed as

$$|\Psi(\bar{\vartheta}, \bar{\varphi})\rangle \rightarrow \frac{\sqrt{N}}{\mathcal{N}} (|v_+\rangle |\psi(\vartheta, \varphi)\rangle + \sqrt{N-1} \cos \frac{\vartheta}{2} |v_-\rangle |0\rangle), \quad (4.2)$$

where  $\mathcal{N} = \sqrt{N^2 \cos^2(\vartheta/2) + N \sin^2(\vartheta/2)}$  is the normalization constant. In Eq. (4.2) we have introduced two orthogonal vectors of  $N-1$  qubits  $|v_{\pm}\rangle$ .

$$\begin{aligned} |v_+\rangle &= \frac{1}{\sqrt{N}} \{ \sqrt{N-1} |N-1,1\rangle + |N-1,0\rangle \}, \\ |v_-\rangle &= \frac{1}{\sqrt{N}} \{ \sqrt{N-1} |N-1,0\rangle - |N-1,1\rangle \}. \end{aligned} \quad (4.3)$$

At the output of the network a projective measurement on the first  $N-1$  qubits is performed in order to determine whether they are in the state  $|v_+\rangle$  or  $|v_-\rangle$ . If the result  $|v_+\rangle$  is obtained, then the  $N$ th qubit is in the desired state  $|\psi(\vartheta, \varphi)\rangle$ . The probability of this outcome is given by

$$P_{|v_+\rangle} = \frac{1}{N \cos^2(\vartheta/2) + \sin^2(\vartheta/2)}. \quad (4.4)$$

This probability is input-state dependent, and it decreases with  $N$ .

There is a difference between this probabilistic process and those considered previously, such as probabilistic cloning [13]. Those only work for set of input states which is finite. The process considered above, however, works for a continuous, and hence infinite, set of input states. It, in fact, works for all input states of the type we are considering. Therefore, we can conclude that the range of applicability of probabilistic devices depends on the process being considered.

## V. CONCLUSION

We have considered a number of different methods of extracting an unknown state from an entangled state formed from that state and a known state. Measuring the state is, as expected, the least effective method. In the case of quantum devices, the universal device was not best one, at least if average fidelity is used as the criterion. Probabilistic quantum devices were seen to work very well for this operation in that they can be used for the entire set of input states.

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## APPENDIX: PROOF OF OPTIMALITY

Let us consider the optimal quantum disentangler which acts as close as possible to the ideal transformation (1.7). The disentangler maps the space spanned by the vectors  $|N;0\rangle$  and  $|N;1\rangle$ , into the space spanned by  $|N-1;0\rangle|1\rangle$  and  $|N-1,0\rangle|1\rangle$ . This suggests that we consider a transformation of the following form:

$$\begin{aligned} |N;0\rangle|d_0\rangle &\rightarrow |N-1;0\rangle(|0\rangle|D_1\rangle + |1\rangle|D_2\rangle), \\ |N;1\rangle|d_0\rangle &\rightarrow |N-1;0\rangle(|0\rangle|D_3\rangle + |1\rangle|D_4\rangle), \end{aligned} \quad (\text{A1})$$

where  $|d_0\rangle$  is the initial state of the disentangler which is supposed to be the same for all inputs and  $|D_j\rangle$  ( $j = 1, \dots, 4$ ) are some unnormalized disentangler state vectors. Our task is to determine these vectors.

Unitarity immediately implies that

$$\begin{aligned} \|D_1\|^2 + \|D_2\|^2 &= 1, \\ \|D_3\|^2 + \|D_4\|^2 &= 1, \\ \langle D_1|D_3\rangle + \langle D_2|D_3\rangle &= 0. \end{aligned} \quad (\text{A2})$$

We shall now use our disentangler transformations (A1) to calculate the fidelity of the actual output to the ideal output (1.7) The input of the disentangler is given by Eq. (1.3). If we introduce a notation  $\bar{\alpha} = \cos(\vartheta/2)$  and  $\bar{\beta} = e^{i\varphi}\sin(\vartheta/2)$  we can write the result of the transformation (A1)

$$\begin{aligned} |\Psi_{out}\rangle &= |N-1;0\rangle \otimes [\bar{\alpha}(|0\rangle|D_1\rangle + |1\rangle|D_2\rangle) + \bar{\beta}(|0\rangle|D_3\rangle \\ &\quad + |1\rangle|D_4\rangle)]. \end{aligned} \quad (\text{A3})$$

We now use this expression to find the output density matrix and trace out the disentangler itself. We define the  $N$ -qubit output density matrix to be

$$\rho_{out} = \text{Tr}_{\text{disentangler}}(|\Psi_{out}\rangle\langle\Psi_{out}|). \quad (\text{A4})$$

The output fidelity is given by

$$\mathcal{F} = \langle\Psi_{ideal}|\rho_{out}|\Psi_{ideal}\rangle, \quad (\text{A5})$$

where  $|\Psi_{ideal}\rangle$  is given by Eq. (1.7). If we denote  $\alpha = \cos(\vartheta/2)$  and  $\beta = e^{i\varphi}\sin(\vartheta/2)$  we can express this fidelity as

$$\begin{aligned} \mathcal{F} &= \frac{1}{(N|\alpha|^2 + |\beta|^2)} \{N|\alpha|^4\|D_1\|^2 + |\beta|^4\|D_4\|^2 \\ &\quad + |\alpha|^2|\beta|^2[\|D_3\|^2 + N\|D_2\|^2 + \sqrt{N}(\langle D_4|D_1\rangle \\ &\quad + \langle D_1|D_4\rangle)] + \alpha^*\beta|\alpha|^2(\sqrt{N}\langle D_1|D_3\rangle + N\langle D_2|D_1\rangle) \\ &\quad + \alpha\beta^*|\alpha|^2(\sqrt{N}\langle D_3|D_1\rangle + N\langle D_1|D_2\rangle) \\ &\quad + \alpha^*\beta|\beta|^2(\sqrt{N}\langle D_2|D_4\rangle + \langle D_4|D_3\rangle) \\ &\quad + \alpha\beta^*|\beta|^2(\sqrt{N}\langle D_4|D_2\rangle + \langle D_3|D_4\rangle) \\ &\quad + (\alpha^*)^2\beta^2\sqrt{N}\langle D_2|D_3\rangle + \alpha^2(\beta^*)^2\sqrt{N}\langle D_3|D_2\rangle\}. \end{aligned} \quad (\text{A6})$$

From this point on we will study two separate cases. First, we will prove optimality of the universal disentangler and then the optimality of the state-dependent disentangler.

### 1. Universal disentangler

Demanding that the fidelity be independent of phases of  $\alpha$  and  $\beta$  we find that

$$\begin{aligned} \sqrt{N}\langle D_1|D_3\rangle + N\langle D_2|D_1\rangle &= 0, \\ \langle D_3|D_2\rangle &= 0, \end{aligned} \quad (\text{A7})$$

$$\sqrt{N}\langle D_2|D_4\rangle + N\langle D_4|D_3\rangle = 0.$$

Assuming these conditions to be satisfied the fidelity becomes

$$\begin{aligned} \mathcal{F} &= \frac{1}{(N|\alpha|^2 + |\beta|^2)} \{N|\alpha|^4\|D_1\|^2 + |\beta|^4\|D_4\|^2 \\ &\quad + |\alpha|^2|\beta|^2[\|D_3\|^2 + N\|D_2\|^2 + \sqrt{N}(\langle D_4|D_1\rangle \\ &\quad + \langle D_1|D_4\rangle)]\}. \end{aligned} \quad (\text{A8})$$

In order for this to be independent of  $\alpha$  and  $\beta$ , the term in brackets must be proportional to

$$(N|\alpha|^2 + |\beta|^2) = N|\alpha|^4 + (N+1)|\alpha|^2|\beta|^2 + |\beta|^4. \quad (\text{A9})$$

Comparing Eqs. (A8) and (A9) we find that

$$\|D_1\| = \|D_4\|,$$

$$(N+1)\|D_4\|^2 = \|D_3\|^2 + N\|D_2\|^2 + \sqrt{N}(\langle D_4|D_1\rangle + \langle D_1|D_4\rangle). \quad (\text{A10})$$

Combining these requirements with those imposed by unitarity we conclude that

$$\|D_3\|^2 = \|D_2\|^2 = 1 - \|D_4\|^2, \quad (\text{A11})$$

and  $\mathcal{F} = \|D_4\|^2$ . This means that in order to maximize  $\mathcal{F}$ , we must maximize  $\|D_4\|^2$ .

Our first step in accomplishing this is to note that by combining the results of Eqs. (A10) and (A11) we have that

$$(N+1) + 2\sqrt{N}x\|D_4\|^2 = 2(N+1)\|D_4\|^2, \quad (\text{A12})$$

where

$$x = \frac{\langle D_4|D_1\rangle + \langle D_1|D_4\rangle}{2\|D_4\|^2}, \quad (\text{A13})$$

and  $-1 \leq x \leq 1$ . Solving for  $\|D_4\|^2$  we find that

$$\|D_4\|^2 = \frac{N+1}{2(N-\sqrt{2}x)}, \quad (\text{A14})$$

which, assuming  $N \geq 2$ , is greatest when  $x = 1$ . This implies that  $|D_1\rangle = |D_4\rangle$  and that

$$\begin{aligned} \|D_4\|^2 &= \frac{N+1}{2(N+1-\sqrt{N})}, \\ \|D_3\|^2 = \|D_2\|^2 &= \frac{N+1-2\sqrt{N}}{2(N+1-\sqrt{N})}. \end{aligned} \quad (\text{A15})$$

Imposing now the conditions on inner products we find that

$$\langle D_3|D_4\rangle = \langle D_2|D_4\rangle = 0. \quad (\text{A16})$$

We can summarize our results in the following way. Let  $\{d_j|j=1,2,3\}$  be a set of three orthonormal vectors and define two parameters  $\gamma_N$  and  $\delta_N$  given by Eq. (3.3) we then have that

$$\begin{aligned} |D_4\rangle &= |D_1\rangle = \gamma_N|d_1\rangle, \\ |D_2\rangle &= \delta_N|d_2\rangle, \\ |D_3\rangle &= \delta_N|d_3\rangle, \end{aligned} \quad (\text{A17})$$

and the universal *optimal* disentangler transformation is given explicitly by Eq. (3.2).

## 2. Input-state dependent disentglers

In order to find the optimal input-state dependent disentangler we find the explicit form of the transformation (A1) such that the *averaged* fidelity  $\bar{\mathcal{F}} = \int d\Omega \mathcal{F}$  [with  $\mathcal{F}$  given by Eq. (A6)] is maximized. Here, as usually, the integration measure is  $d\Omega = \sin \vartheta d\vartheta d\varphi / 4\pi$ . Therefore after the integral over the phase  $\varphi$  is performed we can write the average fidelity as

$$\begin{aligned} \bar{\mathcal{F}} &= \frac{1}{2} \{ \xi_1 N \|D_1\|^2 + \xi_2 \|D_4\|^2 + \xi_3 [\|D_3\|^2 + N\|D_2\|^2] \\ &\quad + \sqrt{N}(\langle D_1|D_4\rangle + \langle D_4|D_1\rangle) \} \end{aligned} \quad (\text{A18})$$

with

$$\begin{aligned} \xi_1 &= \int_0^\pi \frac{\sin \vartheta d\vartheta}{N \cos^2(\vartheta/2) + \sin^2(\vartheta/2)} \cos^4 \frac{\vartheta}{2}, \\ \xi_2 &= \int_0^\pi \frac{\sin \vartheta d\vartheta}{N \cos^2(\vartheta/2) + \sin^2(\vartheta/2)} \sin^4 \frac{\vartheta}{2}, \\ \xi_3 &= \int_0^\pi \frac{\sin \vartheta d\vartheta}{N \cos^2(\vartheta/2) + \sin^2(\vartheta/2)} \sin^2 \frac{\vartheta}{2} \cos^2 \frac{\vartheta}{2}. \end{aligned} \quad (\text{A19})$$

After the integration over the parameter  $\vartheta$  we find

$$\begin{aligned} \xi_1 &= \frac{3 - 4N + N^2 + 2 \ln N}{(N-1)^3}, \\ \xi_2 &= \frac{-1 + 4N - 3N^2 + 2N^2 \ln N}{(N-1)^3}, \\ \xi_3 &= \frac{-1 + N^2 - 2N \ln N}{(N-1)^3}. \end{aligned} \quad (\text{A20})$$

From the unitarity of the disentangling transformation it follows that  $\|D_2\|^2 = 1 - \|D_1\|^2$  and  $\|D_3\|^2 = 1 - \|D_4\|^2$ . When we introduce the notation

$$u = \frac{\langle D_4|D_1\rangle + \langle D_1|D_4\rangle}{2\|D_1\|\|D_4\|}, \quad (\text{A21})$$

where  $-1 \leq u \leq 1$ , and  $\eta_1 = \|D_1\|^2$ ;  $\eta_4 = \|D_4\|^2$  we can rewrite the average fidelity (A18) as

$$\begin{aligned} \bar{\mathcal{F}} &= \frac{1}{2} [ \eta_1 N (\xi_1 - \xi_3) + \eta_4 (\xi_2 - \xi_3) + 2\sqrt{N} \xi_3 u \sqrt{\eta_1 \eta_4} \\ &\quad + \xi_3 (1+N) ]. \end{aligned} \quad (\text{A22})$$

Taking into account that  $\xi_1 > \xi_3$  and  $\xi_2 > \xi_3$  we easily find that the maximum of the mean fidelity (A22) is achieved for  $u = 1$  and  $\eta_1 = \eta_4 = 1$ . In this case we rewrite Eq. (A22) as

$$\mathcal{F} = \frac{1}{2} [\xi_1 N + \xi_2 + 2\sqrt{N}\xi_3]. \quad (\text{A23})$$

When we substitute into Eq. (A23) the explicit expression for the parameters  $\xi_j$  given by Eq. (A20) we find that the mean fidelity is equal to the function  $f_N$  given by Eq. (2.7). This exactly is equal to the mean fidelity of the input-state

disentanglement performed via the state swapping transformation described by Eq. (3.7). In fact, from our conditions  $\eta_1 = \eta_4 = 1$  it directly follows that  $\|D_2\|^2 = \|D_3\|^2 = 0$  while  $\|D_1\|^2 = \|D_4\|^2 = 1$ . In addition, from  $u = 1$  it follows that  $|D_1\rangle = |D_4\rangle$ , so that the optimal state-dependent disentangling transformation is indeed equal to Eq. (3.7), which we wanted to prove.

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