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Quantum statistics of grey-body radiation

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Abstract

We present a microscopic model for a grey body which consists of a blackbody at the temperature T_b surrounded by a semitransparent mirror. We derive the density operator of the grey-body radiation in the photon number or Wigner representation. These relations involve the density matrix or the Wigner function of the incident radiation and kernels which contain information about the blackbody temperature and the mirror. © 1998 Published by Elsevier Science B.V.

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Scattering of radiation off a black hole [1] is a central problem in astrophysics and quantum gravity. Remote sensing [2] is an important issue of applied physics. The concept of grey-body radiation is important to both of them. Grey-body radiation emerges from a physical object with a surface that absorbs and reflects part of the incident radiation. Despite its importance and many detailed investigations [3–6] no complete quantum theory of grey-body radiation has been developed. In this Letter we present the first microscopic theory. We give the density operator $\hat{\rho}^{(g)}$ of the grey-body radiation, expressed either in terms of the Wigner function [7] $W^{(g)}(q,p)$ defined in phase space by the dimensionless quadrature variables q and p, or the density matrix $\rho^{(g)}_{mm'}$ in the photon number representation. We show that three physical factors determine the complete quantum state of the grey-body radiation: (i) the temperature T_b of the body, (ii) the optical properties of the grey body, and (iii) the input state.

When light shines on any body it will in general reemit radiation. A blackbody is characterized by the property that the radiation in the outgoing mode is in a thermal state independent of the quantum state of the incident radiation. The emitted field depends only on the temperature T_b of the body. In this sense blackbody radiation is universal. In contrast, for a grey body the state of the outgoing mode not only depends on the temperature but also on the quantum state of the incident field. It is therefore not universal. To understand these features we need a model that shows how to connect the incident with the outgoing radiation.

In our model we consider for the sake of simplicity only two modes of the radiation field characterized by their wave vectors as shown by the left side of Fig. 1. The first mode with density operator $\hat{\rho}^{(i)}$ corresponds to the radiation incident on the grey body, whereas the second mode is characterized by a field with blackbody radiation $\hat{\rho}^{(b)}$ of temperature T_b. We couple these two modes by, for example, a dielectric

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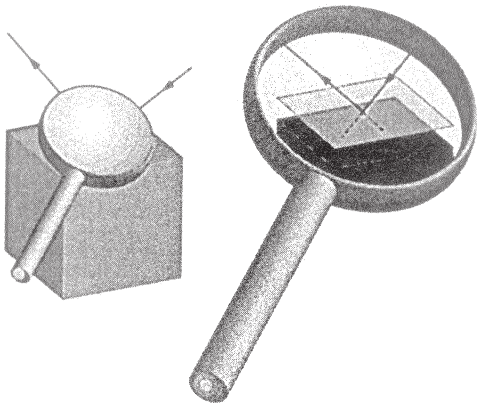

Fig. 1. Microscopic model for the grey-body radiation. Left: We consider two modes characterized by wave vectors corresponding to radiation incident on and scattered off the body. Right: Transformation of the incident into the outgoing radiation. The field of the incident mode is coupled to a mode with thermal radiation from a blackbody at temperature T_b. A dielectric medium acting as a beam splitter provides this coupling. Therefore the radiation emerging from the grey body carries properties of both the incident and the blackbody radiation.

medium acting as a beam splitter as indicated by the right side of Fig. 1. We emphasize that the dielectric medium serves only as a simple representation of much more complicated physical processes. For example, in the case of electromagnetic radiation scattering off a black hole the event horizon serves as a beam splitter at which in-going radiation gets partially reflected or transmitted. The corresponding transmission and reflecting coefficients follow from the appropriate boundary conditions of Maxwell's equations in curved space-time [8]. A similar interpretation of the beam splitter holds true in the context of a remote sensor. The resulting two-mode field with density operator $\hat{\rho}(\tau)$ reads

$$\hat{\rho}(\tau) = \hat{U}(\tau)\, \hat{\rho}^{(i)} \otimes \hat{\rho}^{(b)}\, \hat{U}^{\dagger}(\tau), \tag{1}$$

where $\hat{U}(\tau) \equiv \exp(-i\hat{H}_{int}\tau/\hbar)$ denotes the time evolution operator. Here we have assumed that the interaction described by the Hamiltonian $\hat{H}_{int}$ acts during the time τ.

Since the radiation in the incident mode is absorbed by the blackbody we trace over this mode. Hence in this model

$$\hat{\rho}^{(g)} = \mathrm{Tr}_i \left\{ \hat{U}(\tau)\ \hat{\rho}^{(i)} \otimes \hat{\rho}^{(b)}(T_b)\ \hat{U}^{\dagger}(\tau) \right\} \tag{2}$$

is the density operator of the grey-body radiation.

We note the linear relationship between the density operators $\hat{\rho}^{(i)}$ and $\hat{\rho}^{(g)}$ of the incident and the outgoing radiation. This property is independent of the specific form of the time evolution operator $\hat{U}$ and of the density operator $\hat{\rho}^{(i)}$ of the incident field. It is therefore a universal feature which results from the linearity of quantum mechanics. However, this is the only universal property of the grey-body radiation. From Eq. (2) we observe that in our model the density operator of the grey-body radiation does – apart from the temperature T_b of the body – depend on the specific interaction Hamiltonian $\hat{H}_{int}$ and the full density operator $\hat{\rho}^{(i)}$ of the incident radiation. This stands out most clearly in the photon number representation,

$$\rho^{(g)}_{nn'} \equiv \langle n|\hat{\rho}^{(g)}|n'\rangle = \sum_{m,m'} K^{mm'}_{nn'}\ \rho^{(i)}_{mm'}, \tag{3}$$

following from Eq. (2). Here the summation kernel

$$K^{mm'}_{nn'} \equiv \sum_{k,l} \rho^{(b)}_{kk}\ U^{mk}_{nl}\ (U^{m'k}_{n'l})^*$$

contains the matrix elements $U^{mk}_{nl} \equiv \langle n_b, l_i|\hat{U}|m_i, k_b\rangle$ and $\rho^{(b)}_{kk} \equiv \langle k_b|\hat{\rho}^{(b)}|k_b\rangle$ in the photon number basis $|m_i, k_b\rangle$ of the incident and the blackbody radiation. Hence the specific form of the interaction Hamiltonian determines the matrix elements U^{mk}_{nl} and therefore the kernel. Moreover, Eq. (3) suggests that coherence properties of $\hat{\rho}^{(i)}$ can reflect themselves in $\hat{\rho}^{(g)}$. In the present paper we consider the case of a linear coupler and show that indeed the density matrix of the grey-body radiation is in general not diagonal in the photon number representation.

In our model we assume that the atoms of the medium interact with the radiation field via one-photon processes which give rise to a linear coupling between the incident field mode and the mode of the blackbody radiation. This yields the Hamiltonian [9–12]

$$\hat{H}_{\rm int} = {\rm i}\hbar\lambda(\hat{a}_{\rm b}^\dagger\hat{a}_{\rm i} - \hat{a}_{\rm b}\hat{a}_{\rm i}^\dagger), \tag{4}$$

where $\hat{a}_{\rm i}$ ($\hat{a}_{\rm i}^\dagger$) and $\hat{a}_{\rm b}$ ($\hat{a}_{\rm b}^\dagger$) are the annihilation (creation) operators for the incident and blackbody modes, respectively. Here the interaction strength is denoted by λ. When we write the annihilation operators

$$\hat{a}_j = \frac{1}{\sqrt{2}}(\hat{q}_j + {\rm i}\hat{p}_j)$$

for j = i and b in terms of the dimensionless quadrature operators $\hat{q}_j$ and $\hat{p}_j$ we can identify the evolution operator with

$$\hat{U}(\tau) = \exp(-{\rm i}\hat{H}_{\rm int}\tau/\hbar) = \exp(-{\rm i}\lambda\tau\hat{L}_z),$$

where $\hat{L}_z \equiv \hat{q}_{\rm i}\hat{p}_{\rm b} - \hat{p}_{\rm i}\hat{q}_{\rm b}$ is the z component of the angular momentum operator. Hence $\hat{U}$ describes a rotation in the $q_{\rm i}$–$q_{\rm b}$ plane about the angle $\lambda\tau$. When it acts on a two-mode position eigenstate $|x, y\rangle$ we obtain

$$\exp(-{\rm i}\lambda\tau\hat{L}_z)|x, y\rangle = |tx - ry, rx + ty\rangle, \tag{5}$$

where $t \equiv \cos(\lambda\tau)$ and $r \equiv \sin(\lambda\tau)$ are interpreted as the probability amplitudes of the transmission and the reflection of light of the beam splitter, respectively [9].

Relation (5) allows us to express the two-mode Wigner function

$$\begin{aligned} &W^{(\rm out)}(q_{\rm i}, p_{\rm i}; q_{\rm b}, p_{\rm b}) \\ &\quad = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}{\rm d}y_{\rm i}\int_{-\infty}^{\infty}{\rm d}y_{\rm b}\,\langle q_{\rm i} - y_{\rm i}/2, q_{\rm b} - y_{\rm b}/2| \\ &\quad \times \hat{\rho}^{(\rm out)}|q_{\rm i} + y_{\rm i}/2, q_{\rm b} + y_{\rm b}/2\rangle\, {\rm e}^{{\rm i}(p_{\rm i}y_{\rm i} + p_{\rm b}y_{\rm b})} \end{aligned} \tag{6}$$

of the output fields as a product of the Wigner functions $W^{(\rm b)}$ and $W^{(\rm i)}$ of the blackbody and the incident radiation, respectively. Indeed, by substituting the density operator (1) into the Wigner function (6) and by making use of the transformation property (5) we find [13] the relation

$$\begin{aligned} W^{(\rm out)}(q_{\rm i}, p_{\rm i}; q_{\rm b}, p_{\rm b}) &= W^{(\rm b)}(tq_{\rm b} - rq_{\rm i},\, tp_{\rm b} - rp_{\rm i}) \\ &\times W^{(\rm i)}(tq_{\rm i} + rq_{\rm b},\, tp_{\rm i} + rp_{\rm b}), \end{aligned} \tag{7}$$

In our model of the grey body we only observe the blackbody mode. We therefore have to integrate the two-mode Wigner function $W^{(\rm out)}$, Eq. (7), over the pair $(q_{\rm i}, p_{\rm i})$ of phase-space variables associated with the incident mode. Hence, the Wigner function of the grey-body radiation reads

$$W^{(\rm g)}(q_{\rm b}, p_{\rm b}) = \int_{-\infty}^{\infty}{\rm d}q_{\rm i}\int_{-\infty}^{\infty}{\rm d}p_{\rm i}\, W^{(\rm out)}(q_{\rm i}, p_{\rm i}; q_{\rm b}, p_{\rm b}). \tag{8}$$

We now rewrite this integral taking into account the explicit form

$$W^{(\rm b)}(q, p) = \frac{1}{\pi(1 + 2\bar{n}_{\rm b})}\exp\left(-\frac{q^2 + p^2}{1 + 2\bar{n}_{\rm b}}\right) \tag{9}$$

of the Wigner function of the thermal field. Here $\bar{n}_{\rm b} = [\exp(\hbar\omega/k_{\rm B}T_{\rm b}) - 1]^{-1}$ is the mean photon number of a thermal state at frequency ω and temperature $T_{\rm b}$. The Boltzmann constant is denoted by $k_{\rm B}$.

We insert Eq. (9) into Eqs. (7) and (8), perform the substitution of variables $q_{\rm i} = (q' - rq_{\rm b})/t$ and $p_{\rm i} = (p' - rp_{\rm b})/t$, and obtain the Wigner function

$$\begin{aligned} W^{(\rm g)}(q, p) &= \int_{-\infty}^{\infty}{\rm d}q'\int_{-\infty}^{\infty}{\rm d}p' \\ &\times \mathcal{K}(q, p; q', p')\, W^{(\rm i)}(q', p') \end{aligned} \tag{10}$$

of the grey-body radiation. Here the integral kernel

$$\begin{aligned} \mathcal{K}(q, p; q', p') &\equiv \frac{1}{\pi t^2(1 + 2\bar{n}_{\rm b})} \\ &\times \exp\left(-\frac{(q - rq')^2 + (p - rp')^2}{t^2(1 + 2\bar{n}_{\rm b})}\right) \end{aligned} \tag{11}$$

is normalized to unity.

Eq. (10) expresses the Wigner function of the grey-body radiation in terms of the Wigner function of the incident radiation. The integral kernel $\mathcal{K}$, Eq. (11), depends only on the temperature of the body and the properties of the beam splitter characterized by the coefficents r and t. In particular, the kernel has the limits

$$\mathcal{K}(q,p;q',p') = W^{(\mathrm{b})}(q,p), \qquad \text{for } t^2=1,$$
$$= \delta(q-q')\,\delta(p-p') \quad \text{for } t^2=0. \tag{12}$$

which shows that in the case of a completely transparent mirror, that is $t^2 = 1$, the incident radiation is completely thermalized and the grey body radiates thermal radiation described by the Wigner function (9). Hence the grey body operates as a blackbody. On the other hand, in the case of a perfectly reflecting mirror, that is $t^2 = 0$, the grey-body radiation is the incident radiation described by the Wigner function $W^{(\mathrm{i})}(q,p)$.

We now illustrate relation (10) further using either a thermal state or a coherent state as the incident field. We start our discussion with the case when the incident radiation is in a thermal state at temperature T_{i} with mean photon number $\bar{n}_{\mathrm{i}}$. We substitute the corresponding Wigner function in Eq. (10) and perform the integration. We find that the grey-body radiation is in the thermal state with the mean photon number $\bar{n}_{\mathrm{g}} = t^2\bar{n}_{\mathrm{b}} + r^2\bar{n}_{\mathrm{i}}$. This result proves one of the key assumptions made in Ref. [3].

However, Eq. (10) is much more general. To demonstrate this and to bring out most clearly that the grey-body radiation can carry phase information, we now analyze the second example in which the incident field is in a coherent state described by

$$W^{(\mathrm{i})}_{\mathrm{coh}}(q,p) = \frac{1}{\pi}\exp[-(q-\bar{q})^2-(p-\bar{p})^2], \tag{13}$$

where $\bar{q}$ and $\bar{p}$ are the mean values of the quadrature operators. We substitute Eq. (13) into Eq. (10) and obtain the Wigner function

$$W^{(\mathrm{g})}_{\mathrm{coh}}(q,p) = \frac{1}{\pi(1+2t^2\bar{n}_{\mathrm{b}})} \times \exp\left(-\frac{(q-r\bar{q})^2+(p-r\bar{p})^2}{1+2t^2\bar{n}_{\mathrm{b}}}\right) \tag{14}$$

of the grey-body radiation [1] when the incident field is in a coherent state. Note that $W^{(\mathrm{g})}$ is still a Gaussian located away from the origin of phase space and hence the phase information of the incident coherent field is partially preserved in the grey-body radiation. This example clearly shows that the grey-body radiation cannot be ad hoc expressed only as a statistical mixture of Fock states.

To illustrate this feature from a different point of view we now return to the connection formula (3) in the photon number representation. We explicitly calculate the summation kernel $K^{mm'}_{nn'}$ by making use of the relations [15]

$$\rho^{(\mathrm{g})}_{nn'} = 2\pi\int_{-\infty}^{\infty}\mathrm{d}q\int_{-\infty}^{\infty}\mathrm{d}p\, W^{(\mathrm{g})}(q,p)\, W_{n'n}(q,p), \tag{15}$$

$$W^{(\mathrm{i})}(q',p') = \sum_{m,m'} W_{mm'}(q',p')\,\rho^{(\mathrm{i})}_{mm'} \tag{16}$$

with [15,16]

$$W_{nn'}(q,p) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}y\,\langle q-y/2|n\rangle\langle n'|q+y/2\rangle\,\mathrm{e}^{\mathrm{i}py}$$
$$= \frac{(-1)^{n'}}{\pi}\left(\frac{n'!}{n!}\right)^{1/2}[\sqrt{2}(q-\mathrm{i}p)]^{n-n'} \times \exp(-q^2-p^2)\,L^{(n-n')}_{n'}(2q^2+2p^2), \tag{17}$$

where $L^{(m)}_n(x)$ is a generalized Laguerre polynomial. When we substitute Eq. (10) into Eq. (15) and then use Eq. (16) we find by comparison with Eq. (3) the formula

$$K^{mm'}_{nn'} = 2\pi\int\mathrm{d}q\int\mathrm{d}p\int\mathrm{d}q'\int\mathrm{d}p'\,\mathcal{K}(q,p;q',p') \times W_{n'n}(q,p)\,W_{mm'}(q',p'). \tag{18}$$

We obtain an explicit form of the summation kernel K by substituting the integration kernel $\mathcal{K}$, Eq. (11), and expression (17) for $W_{nn'}$ into Eq. (18). The integrals can be calculated analytically and we arrive [17] at

$$K^{mm'}_{nn'} = \delta_{n-n',m-m'}\left(\frac{n'!m'!}{n!m!}\right)^{1/2}\frac{(n+m')!}{n'!\,m'!} \times r^{n-n'}\,t^{2(n'+m')}\frac{(1+\bar{n}_{\mathrm{b}})^{m'}\bar{n}_{\mathrm{b}}^{n'}}{(1+t^2\bar{n}_{\mathrm{b}})^{n+m'+1}} \times {}_2F_1(-n',-m';-m'-n;z), \tag{19}$$

where ${}_2F_1(\alpha,\beta;\gamma;z)$ is the hypergeometric function with the argument $z = 1-r^2/t^4\bar{n}_{\mathrm{b}}(1+\bar{n}_{\mathrm{b}})$.

[1] The P-function of this particular state was first calculated by Lachs [14] in a different context, as a mixture of thermal and coherent state.

It is because of the Kronecker delta in Eq. (19) that the photon statistics $P_n^{(g)} \equiv \langle n|\hat{\rho}^{(g)}|n\rangle$ and $P_m^{(i)} \equiv \langle m|\hat{\rho}^{(i)}|m\rangle$ of the grey-body radiation and of the incident radiation are simply connected according to Eq. (3) by

$$P_n^{(g)} = \sum_m K_{nn}^{mm} P_m^{(i)}. \tag{20}$$

Thus, if the incident radiation is a statistical mixture of photon number states then Eq. (20) contains the complete information because the density matrices have no off-diagonal elements. However, as soon as the incident radiation contains coherences, described by the off-diagonal elements of the density matrix, one needs to take into account the off-diagonal elements of the grey-body radiation as implied by Eq. (3).

In conclusion, we have presented a microscopic model of the grey body which consists of a blackbody at the temperature T_b and a beam splitter characterized by the amplitudes of transmission and reflection. The beam splitter is modeled as a linear coupler between the blackbody radiation and the incident radiation. We have derived general expressions, Eqs. (3) and (10), for the density operator of the grey-body radiation in the photon number or Wigner representation. In our discussion we have considered the ideal mirror, i.e. the lossless mirror which does not introduce additional noise or losses. Moreover we have focused on two modes only. However, we emphasize that the present treatment can be generalized[2] leading to more complicated kernels.

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[2] It is straightforward to generalize our model to the case of a nonideal mirror. In this case expressions (3) and (10) preserve their form; the kernels, however, take a new form. Indeed when the losses are due to a Gaussian reservoir the new integral kernel can be expressed as the convolution of the old one with the corresponding coarsening Gaussian function.