

## Parametrized discrete phase-space functions

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Using discrete displacement-operator expansion,  $s$ -parametrized phase-space functions associated with the operators in a finite-dimensional Hilbert space are introduced and their properties are studied. In particular, the phase-space functions associated with the density operator can be regarded as quasidistributions whose properties are similar to those of the well-known quasidistributions in the continuous phase space. So the  $Q$  function ( $s = -1$ ) is non-negative and can be measured directly in particular experiments, whereas the  $P$  function ( $s = 1$ ) corresponds to the diagonal form of the density operator in an overcomplete basis. Except for the  $W$  function ( $s = 0$ ), the introduction of discrete phase-space functions requires the choice of a special reference state. We finally present a simple model for measuring the discrete  $Q$  function. [S1050-2947(96)07606-8]

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### I. INTRODUCTION

In quantum mechanics, the state of a system is usually described by means of a wave function or, more generally, of a density operator. This description contains all the information we can get from the system. However, there are also other possible ways to describe the system—using the so-called quasidistributions. These functions are defined on the phase space of the system—and one of their advantages is that they can be directly compared to the classical stochastic distribution functions. There are, however, differences with respect to the classical phase-space description, originated from the noncommuting nature of the conjugated quantities. The first quasidistribution—the Wigner  $W$  function [1,2]—generates the proper marginal distributions of the position and momentum and of any of their linear combinations; in comparison to the classical distribution functions it can also take negative values. The Glauber-Sudarshan  $P$  function [3] enables us to write the density operator in a diagonal form in an overcomplete basis of the coherent states. Its disadvantage is, however, that it is often highly singular and not well behaved. The Husimi  $Q$  function [4] is the most similar to the classical probability distribution functions, because it is always non-negative; moreover this function can be directly measured in experiments. However, it is too “smeared” so that some information (like the quantum interference) is difficult to read immediately from it. These quasidistributions also yield a very effective tool for calculating moments of physical quantities: The  $P$  function enables a direct calculation of means of normally ordered creation and annihilation operators, the  $Q$  function that of the antinormally ordered operators, and the  $W$  function of the symmetrically ordered operators, e.g., of the position, momentum, and of their functions. These three quasidistributions are, however, not the only ones: Cahill and Glauber [5] showed that the  $W$ ,  $Q$ , and the  $P$  functions can be treated as special cases of an  $s$ -parametrized class of functions. The  $P$  function then corresponds to the value  $s = 1$ , the  $W$  function to  $s = 0$ , and the

$Q$  function to  $s = -1$ ; the parameter  $s$  can also take any other real values. Other generalizations can be found, e.g., in the Gardiner book [6] or in the recent work of Wünsche [7], who suggested a complete class of Gaussian quasidistributions, parametrized by a three-dimensional complex vector, and offered a very clear mathematical formalism for them. These functions can be connected not only to the density operators but to any other operator of the Hilbert space. In this case we speak about phase-space functions rather than quasidistributions.

Until recently these functions were treated only for the continuous phase spaces, corresponding to the systems with an infinite-dimensional Hilbert space. However, there are many quantum systems which are profitably modeled by means of finite-dimensional Hilbert spaces: e.g., the spin systems, several-level atoms in quantum optics, electrons on molecules with a finite number of sites, etc. Today these systems are again very intensively studied, e.g., with their connection to quantum cryptography [8], quantum teleportation, and superdense coding [9], or the quantum computation [10]. Also the possibilities to construct an arbitrary state of an optical field with a limited photon number (see, e.g., [11]), or to reconstruct the density matrix of such a system (e.g., [12]) are of current interest.

To our knowledge the first who considered the  $W$  function for discrete systems were Wootters [13], Galetti and de Toledo Piza [14], and Cohendet *et al.* [15]. Their idea was based on the modulo  $N$  algebra of the discrete quantities ( $N$  being the dimension of the Hilbert space), which means that the phase space for the  $W$  function is a set of points on a torus. Such  $W$  functions then have the meaning that their marginal sums over the generalized “lines” are probabilities of particular values of some physical quantities. This approach was used, e.g., for calculating the number-phase  $W$  function in the Pegg-Barnett model in [16] and [17]. A different approach was suggested, e.g., in [18–21] where the phase space was considered to be a sphere and where the  $W$  functions for various spin or atomic states were calcu-

lated. Here the relation to a description of measurement procedures was via calculating an overlap of this function with a function corresponding to the measured quantity. Quite recently Leonhardt [22,23] suggested a scheme for reconstruction of the discrete  $W$  function from the marginal distributions—the discrete quantum state tomography which brought this discrete phase-space formalism even closer to physical measurements. He also offered a way for treating the  $W$  functions for systems with even-dimensional Hilbert spaces introducing a discrete phase space with both integer and half-odd coordinates. The discrete  $Q$  function was proposed in [24] and [25]; the method was based on the operational definitions of propensities introduced by Wódkiewicz [26] and anticipated by Aharonov *et al.* [27]. Recently, a similar phase-space function was suggested also by Galetti and Marchiolli [28]. The  $Q$  function can be defined as the probability of the system to be in some coherent state; here the definition of the coherent states is in the Perelomov sense [29], i.e., an overcomplete set of displacements of some reference state (which is also called the “quantum ruler”). In [24,25] it was shown how to interpret this function as a set of measurements and how to reconstruct the original density matrix. A deep survey of the phase-space methods can be found in the Leonhardt work [23].

The aim of this paper is to give a unified approach to the problem of discrete phase-space functions and quasidistributions. In particular, the known discrete quasidistributions are generalized to a class of parametrized quasidistributions and their properties are studied. In Sec. II the class of  $s$ -parametrized discrete phase-space functions is introduced, their properties are discussed, and special cases of quasidistributions are considered, such as the  $W$  function, the  $Q$  function, and the  $P$  function. In Sec. III some illustrations of these functions are presented and a possible model of measurements of the discrete  $Q$  function is suggested. Some concluding remarks are given in Sec. IV.

## II. PARAMETRIZED PHASE-SPACE FUNCTIONS

Let us consider an  $N$ -dimensional Hilbert space and introduce two orthonormal basis-vector systems  $|u_k\rangle$  ( $U$  basis) and  $|v_l\rangle$  ( $V$  basis),  $k, l = 0, \dots, N-1$ , which are related to each other by a discrete Fourier transformation:

$$|u_k\rangle = \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} \exp\left(-i \frac{2\pi}{N} kl\right) |v_l\rangle, \quad (1)$$

$$|v_l\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \exp\left(i \frac{2\pi}{N} kl\right) |u_k\rangle. \quad (2)$$

Note that the Kronecker symbol  $\delta_{k,l}$  can be written as the sum

$$\delta_{k,k'} = \frac{1}{N} \sum_{l=0}^{N-1} \exp\left(-i \frac{2\pi}{N} (k-k')l\right), \quad (3)$$

where the difference of the arguments,  $k-k'$ , is taken modulo  $N$ . The basis vectors  $|u_k\rangle$  and  $|v_l\rangle$  can be regarded as the eigenvectors of two Hermitian operators  $\hat{U}$  and  $\hat{V}$ , respectively,

$$\hat{U}|u_k\rangle = k|u_k\rangle, \quad (4)$$

$$\hat{V}|v_l\rangle = l|v_l\rangle. \quad (5)$$

Clearly, the operators  $\hat{U}$  and  $\hat{V}$ , which are analogies of position and momentum in the continuous phase space, represent complementary properties, i.e., the squared scalar product  $|\langle u_k|v_l\rangle|^2 = 1/N$  does not depend on the indices  $k, l$ .

### A. Displacement-operator expansion

To develop a phase-space formalism for a finite-dimensional Hilbert space, it is useful to introduce a displacement operator. For this purpose let us consider the operators

$$\hat{R}_u(n) = \exp\left(-i \frac{2\pi}{N} n \hat{V}\right), \quad (6)$$

$$\hat{R}_v(m) = \exp\left(i \frac{2\pi}{N} m \hat{U}\right), \quad (7)$$

which rotate the basis vectors  $|u_k\rangle$  and  $|v_l\rangle$  as

$$\hat{R}_u(n)|u_k\rangle = |u_{k+n}\rangle, \quad (8)$$

$$\hat{R}_v(m)|v_l\rangle = |v_{l+m}\rangle, \quad (9)$$

where the indices  $(k+n)$  and  $(l+m)$  are taken modulo  $N$  (this convention is used throughout the paper). The operators  $\hat{R}_u(n)$  and  $\hat{R}_v(m)$ , which fulfill the so-called Weyl commutation relation [2,30–32]

$$\hat{R}_u(n)\hat{R}_v(m) = \exp\left(-i \frac{2\pi}{N} mn\right) \hat{R}_v(m)\hat{R}_u(n), \quad (10)$$

can be used to define a displacement operator by [33,24,25]

$$\begin{aligned} \hat{D}(n,m) &\equiv \hat{R}_u(n)\hat{R}_v(m) \exp\left(i \frac{\pi}{N} mn\right) \\ &= \hat{R}_v(m)\hat{R}_u(n) \exp\left(-i \frac{\pi}{N} mn\right), \end{aligned} \quad (11)$$

so that

$$\hat{D}^\dagger(n,m) = \hat{D}(-n,-m). \quad (12)$$

From the relations

$$\hat{D}(n,m+N) = \hat{D}(n,m)(-1)^n, \quad (13)$$

$$\hat{D}(n+N,m) = \hat{D}(n,m)(-1)^m, \quad (14)$$

the operator  $\hat{D}(n,m)$  is seen not to satisfy the condition of modulo  $N$  invariance. It is therefore advantageous to work with a displacement operator of the type  $\hat{D}(2n,2m)$ , where the values of  $n$  (and  $m$ ) are the integers modulo  $N$  if  $N$  is odd [e.g., the  $N$  integers in the interval from  $-(N-1)/2$  to  $(N-1)/2$ ], and the integers and half odds modulo  $N$  if  $N$  is even [e.g., the  $2N$  integers and half odds in the interval from  $-N/2$  to  $(N-1)/2$ ]. Note that for even  $N$  the operator

$\hat{D}(2n, 2m)$  with integers  $n$  and  $m$  would not represent all the possible displacements. (Another solution of the problem of modulo  $N$  invariance was given by Galetti and Toledo Piza [33].)

The operator  $\hat{D}(2n, 2m)$  exhibits the following trace properties. If  $N$  is odd, we find that

$$\text{Tr}[\hat{D}(2n, -2m)\hat{D}(2n', -2m')] = N\delta_{n, -n'}\delta_{m, -m'}, \tag{15}$$

which for  $n' = m' = 0$  reduces to

$$\text{Tr}[\hat{D}(2n, -2m)] = N\delta_{n,0}\delta_{m,0}, \tag{16}$$

whereas for even  $N$  we obtain

$$\begin{aligned} \text{Tr}[\hat{D}(2n, -2m)\hat{D}(2n', -2m')] &= N[\delta_{n, -n'} + (-1)^{2m}\delta_{n, -n'+N/2}] \\ &\times [\delta_{m, -m'} + (-1)^{2n}\delta_{m, -m'+N/2}], \end{aligned} \tag{17}$$

and particularly ( $n' = m' = 0$ )

$$\text{Tr}[\hat{D}(2n, -2m)] = N(\delta_{n,0} + \delta_{n, +N/2})(\delta_{m,0} + \delta_{m, +N/2}). \tag{18}$$

The relations (15) and (17) enable us to expand an operator  $\hat{F}$  as

$$\hat{F} = \frac{N}{M} \sum_{m,n} \tilde{f}(m, n) \hat{D}(2n, -2m), \tag{19}$$

where the discrete  $c$ -number function

$$\tilde{f}(m, n) \equiv \frac{1}{M} \text{Tr}[\hat{F}\hat{D}(-2n, 2m)] \tag{20}$$

can be regarded as the characteristic function of  $\hat{F}$ . Here and in the following the sums run in an interval of length  $N$ , over  $M = N$  integers modulo  $N$  if  $N$  is odd, and  $M = 2N$  integers and half integers modulo  $N$  if  $N$  is even. Note that when  $N$  is even, the relations

$$\tilde{f}\left(m, n + \frac{N}{2}\right) = (-1)^{2m}\tilde{f}(m, n), \tag{21}$$

$$\tilde{f}\left(m + \frac{N}{2}, n\right) = (-1)^{2n}\tilde{f}(m, n) \tag{22}$$

are valid, which shows that  $\tilde{f}(m, n)$  has only  $N \times N$  independent elements.

Let us mention some interesting properties of the characteristic functions. Equation (19) together with Eq. (16) or Eq. (18) implies that

$$\text{Tr}(\hat{F}) = M\tilde{f}(0, 0), \tag{23}$$

that is to say, the trace of an operator  $\hat{F}$  is equal to the value of its characteristic function at the point ( $n = 0, m = 0$ ). Using Eqs. (19) and (20) the trace of the product of two operators  $\hat{F}$  and  $\hat{G}$  can be written as

$$\text{Tr}(\hat{F}\hat{G}) = N \sum_{m,n} \tilde{f}(m, n) \tilde{g}(-m, -n), \tag{24}$$

where  $\tilde{g}(m, n)$  is given by Eq. (20), with  $\hat{G}$  in place of  $\hat{F}$ . For operators  $\hat{F}$  which commute with the parity operator  $\hat{P}$ ,  $\hat{F}\hat{P} = \hat{P}\hat{F}$ , where

$$\langle u_k | \hat{P} | u_l \rangle = \delta_{k, -l}, \quad \langle v_r | \hat{P} | v_s \rangle = \delta_{r, -s}, \tag{25}$$

we can prove that

$$\tilde{f}(-m, -n) = \tilde{f}(m, n), \tag{26}$$

whereas for operators anticommuting with the parity operator,  $\hat{F}\hat{P} = -\hat{P}\hat{F}$ , we obtain

$$\tilde{f}(-m, -n) = -\tilde{f}(m, n). \tag{27}$$

For Hermitian operators  $\hat{F} = \hat{F}^\dagger$  it can be shown that

$$\tilde{f}(-m, -n) = \tilde{f}^*(m, n). \tag{28}$$

Hence from Eqs. (26) and (28) we see that Hermitian operators with even parity have real characteristic functions. When  $\tilde{f}(m, n)$  is the characteristic function (20) of an operator  $\hat{F}$ , then the characteristic function  $\tilde{f}_{k,l}(m, n)$  of the displaced operator

$$\hat{F}_{k,l} \equiv \hat{D}(k, l)\hat{F}\hat{D}(-k, -l) \tag{29}$$

is equal to

$$\tilde{f}_{k,l}(m, n) = \tilde{f}(m, n) \exp\left(i \frac{4\pi}{N} (mk + nl)\right). \tag{30}$$

Calculating the characteristic function of an operator  $\hat{F}$  in the  $U$  basis yields

$$\tilde{f}(m, n) = \frac{1}{M} \sum_r \langle u_{r+2n} | \hat{F} | u_r \rangle \exp\left(i \frac{4\pi}{N} m(n+r)\right). \tag{31}$$

(The index  $r$  takes  $N$  integral values.)

In particular, applying Eq. (19) to the density operator  $\hat{\rho}$ , we see that

$$\hat{\rho} = \frac{N}{M} \sum_{m,n} \tilde{W}(m, n) \hat{D}(2n, -2m), \tag{32}$$

where the characteristic function of  $\hat{\rho}$ ,

$$\tilde{W}(m, n) \equiv \frac{1}{M} \text{Tr}[\hat{\rho}\hat{D}(-2n, 2m)], \tag{33}$$

can be regarded as the characteristic function of the quantum state. The definition of  $\tilde{W}(m, n)$  in Eq. (33) is equivalent to the definition given by Leonhardt [22,23], as can be seen from Eq. (31). By means of Eq. (24) the expectation value of an operator  $\hat{F}$  can then be given by

$$\text{Tr}(\hat{\rho}\hat{F}) = N \sum_{m,n} \tilde{W}(m, n) \tilde{f}(-m, -n). \tag{34}$$

Let us now briefly mention some properties of the characteristic function  $\tilde{W}(m,n)$ . The zero point value is associated with the normalization condition, i.e.,  $\tilde{W}(0,0)=1/M$ . The value  $\tilde{W}(1/2,0)$  is related to the mean value of the operator  $\hat{U}$ ,

$$M\tilde{W}\left(\frac{1}{2},0\right)=\left\langle\exp\left(i\frac{2\pi}{N}\hat{U}\right)\right\rangle\equiv r_u\exp(i\bar{\phi}_u). \quad (35)$$

The  $1/2$  in the argument means the number that when multiplied by 2 is equal to 1 mod  $N$ . For  $N$  odd it is  $-(N-1)/2$  if the arguments run between  $-(N-1)/2$  and  $(N-1)/2$ ; for  $N$  even it is just  $1/2$ . In the goniometric decomposition the absolute value  $r_u$  in Eq. (35) is related to the Bandilla-Paul phase uncertainty measure  $\sigma_u^2=1-r_u^2$  [34] which takes values between zero (sharp value of  $\hat{U}$ ) and unity (completely uncertain  $\hat{U}$ ). The argument  $\bar{\phi}_u$  represents a preferred angular orientation of  $\exp(i2\pi\hat{U}/N)$ ; we can define a kind of mean value of  $\hat{U}$  which respects the mod  $N$  properties of this quantity by  $\bar{U}\equiv\bar{\phi}_u N/(2\pi)$  [for the eigenvalues of  $\hat{U}$  chosen symmetrically around zero and the phase window  $(-\pi,\pi)$ ]. A similar relation holds also for the  $\hat{V}$  operator. These quantities are studied in more detail in [35], and the uncertainty relations between uncertainties associated with the operators  $\hat{U}$  and  $\hat{V}$  are discussed in [36].

### B. Phase-space-point operators

Let  $\hat{\rho}_0$  be a density operator of a reference state whose characteristic function  $\tilde{W}_0(m,n)$  has no zero elements. In order to define  $s$ -parametrized phase-space functions, we first introduce two kinds of  $s$ -parametrized phase-space-point operators  $\hat{T}_x(k,l;s)$ ,  $x=\pm 1$ ,

$$\begin{aligned} \hat{T}_x(k,l;s)\equiv & M^{-s-1}\sum_{m,n}\tilde{W}_0^{-s}(m,n)\hat{D}(-2xn,2xm) \\ & \times\exp\left[-i\frac{4\pi}{N}(kxm+lxn)\right]. \end{aligned} \quad (36)$$

Again, the arguments  $m$ ,  $n$ ,  $k$ , and  $l$  comprise all the integers modulo  $N$  if  $N$  is odd [e.g., between  $-(N-1)/2$  and  $(N-1)/2$ ] and all the integers and half odds modulo  $N$  if  $N$  is even [e.g., between  $-N/2$  and  $(N-1)/2$ ]. The pairs  $(k,l)$  thus form a discrete phase space. We see that the phase-space-point operators  $\hat{T}_x(k,l;s)$  are uniquely defined and Hermitian for integer  $s$ . On the other hand, when the function  $\tilde{W}_0(m,n)$  takes negative or complex values, then for noninteger real  $s$  these operators are not defined uniquely. This is due to the arbitrariness in the definition of  $\tilde{W}_0^{-s}(m,n)$  in this case. Nevertheless, we can formally avoid this problem by defining this power as

$$\tilde{W}_0^{-s}(m,n)=|\tilde{W}_0(m,n)|^{-s}\exp\{-is\arg[\tilde{W}_0(m,n)]\}, \quad (37)$$

where the  $(2\pi)$  interval for the values of  $\arg[\tilde{W}_0(m,n)]$  must be fixed for each point  $(m,n)$ . So as to be  $\hat{T}_x$  Hermitian, we require that if in the point  $(m,n)$  the interval is

chosen  $[\alpha,\alpha+2\pi)$ , then in the point  $(-m,-n)$  the interval must be  $(-\alpha-2\pi,-\alpha]$ . This leads to the relation

$$\tilde{W}_0^{-s}(m,n)=[\tilde{W}_0^{-s}(-m,-n)]^*, \quad (38)$$

which ensures that  $\hat{T}_x$  is Hermitian. Note that there is no confusion with the point  $(0,0)$ , where the characteristic function of the reference state is real and positive.

The operators  $\hat{T}_x(k,l;s)$  exhibit a number of interesting properties. For odd  $N$  they are mutually trace orthonormal,

$$\text{Tr}[\hat{T}_x(k,l;s)\hat{T}_{-x}(k',l';-s)]=N\delta_{k,k'}\delta_{l,l'}. \quad (39)$$

For even  $N$  they are not orthogonal, so that the right-hand side of Eq. (39) becomes more complicated; a simple relation can only be obtained for integer  $s$ ,

$$\begin{aligned} \text{Tr}[\hat{T}_x(k,l;s)\hat{T}_{-x}(k',l';-s)] \\ =N[\delta_{k,k'}+(-1)^{2l}\delta_{k-N(s-1)/2,k'}] \\ \times[\delta_{l,l'}+(-1)^{2k}\delta_{l-N(s-1)/2,l'}]. \end{aligned} \quad (40)$$

For odd  $N$  the operators  $\hat{T}_x(k,l;s)$  have unity traces,

$$\text{Tr}[\hat{T}_x(k,l;s)]=1, \quad (41)$$

whereas for even  $N$  the relations

$$\text{Tr}[\hat{T}_x(k,l;s)]=\begin{cases} 2, & k,l \text{ integers} \\ 0, & k \text{ or } l \text{ half odd} \end{cases} \quad (42)$$

are valid. In any case, the operators  $\hat{T}_x(k,l;s)$  resolve the unity,

$$\frac{1}{M}\sum_{k,l}\hat{T}_x(k,l;s)=\hat{I}. \quad (43)$$

When the reference state has even parity,  $\hat{P}\hat{\rho}_0\hat{P}=\hat{\rho}_0$ , then  $\hat{T}_x(k,l;s)=\hat{T}_{-x}(k,l;s)$ . This corresponds to the usual case of continuous phase space, where the reference state is the vacuum state which has even parity and therefore there is only one class of  $s$ -parametrized phase-space-point operators. In this case, these operators are uniquely defined for any real  $s$ , because the characteristic function of the vacuum is a Gaussian, which is always positive. It should be mentioned that the discrete  $s$ -parametrized phase-space-point operators  $T_x(k,l;s)$  are quite similar to the  $s$ -parametrized operator  $\delta$  functions in the continuous case (see, e.g., [37]).

### C. Phase-space functions

The operators  $T_x(k,l;s)$ ,  $x=\pm 1$ , can be used to perform the following operator expansion:

$$\hat{F}=\frac{N}{M}\sum_{k,l}f_x(k,l;s)\hat{T}_{-x}(k,l;-s), \quad (44)$$

where the discrete  $s$ -parametrized  $c$ -number functions

$$f_x(k,l;s)=\frac{1}{M}\text{Tr}[\hat{F}\hat{T}_x(k,l;s)] \quad (45)$$

can be regarded as phase-space functions associated with the operator  $\hat{F}$ . To prove Eq. (44), we write the right-hand side as

$$\begin{aligned} & \frac{N}{M} \sum_{k,l} f_x(k,l;s) \hat{T}_{-x}(k,l;-s) \\ &= \frac{N}{M^2} \sum_{k,l} \text{Tr}[\hat{F} \hat{T}_x(k,l;s)] \hat{T}_{-x}(k,l;-s). \end{aligned} \quad (46)$$

Using Eqs. (36) and (20), and taking into account that

$$\delta_{k,l} = \frac{1}{M} \sum_m \exp\left[-i \frac{4\pi}{N} (k-l)m\right], \quad (47)$$

we arrive at

$$\frac{N}{M} \sum_{k,l} f_x(k,l;s) \hat{T}_{-x}(k,l;-s) = \frac{N}{M} \sum_{m,n} \tilde{f}(m,n) \hat{D}(2n,-2m), \quad (48)$$

which, according to Eq. (19), is equal to  $\hat{F}$ .

The trace of an operator  $\hat{F}$  is given by the sum over all values of an associated phase-space function,

$$\text{Tr}(\hat{F}) = \sum_{k,l} f_x(k,l;s), \quad (49)$$

which follows from (43) and (45). The trace of the product of two operators  $\hat{F}$  and  $\hat{G}$  is equal to the overlap,

$$\text{Tr}(\hat{F}\hat{G}) = N \sum_{k,l} f_x(k,l;s) g_{-x}(k,l;-s), \quad (50)$$

which when  $N$  is odd simply follows from the orthogonality of the  $\hat{T}$  operators, Eq. (39). For even  $N$  we can use Eq. (24) and write (e.g., for  $x=1$ )

$$\begin{aligned} & \sum_{k,l} f_1(k,l;s) g_{-1}(k,l;-s) \\ &= \frac{1}{M^2} \sum_{k,l} \text{Tr}[\hat{F} \hat{T}_1(k,l;s)] \text{Tr}[\hat{G} \hat{T}_{-1}(k,l;-s)] \\ &= \frac{1}{M^4} \sum_{k,l,m,n,m',n'} \left\{ \tilde{W}_0^{-s}(m,n) \tilde{W}_0^s(m',n') \right. \\ & \quad \times \exp\left[-i \frac{4\pi}{N} [k(m-m') + l(n-n')]\right] \\ & \quad \left. \times \text{Tr}[\hat{F} \hat{D}(-2n,2m)] \text{Tr}[\hat{G} \hat{D}(2n',-2m')]\right\}, \end{aligned} \quad (51)$$

where we have used Eqs. (45) and (36). Using the Kronecker  $\delta$  expansion (47) and recalling Eqs. (20) and (24), we arrive at Eq. (50):

$$\begin{aligned} & \sum_{k,l} f_1(k,l;s) g_{-1}(k,l;-s) \\ &= \frac{1}{M^2} \sum_{m,n} \text{Tr}[\hat{F} \hat{D}(-2n,2m)] \text{Tr}[\hat{G} \hat{D}(2n,-2m)] \\ &= \sum_{m,n} \tilde{f}(m,n) \tilde{g}(-m,-n) = \frac{1}{N} \text{Tr}(\hat{F}\hat{G}). \end{aligned} \quad (52)$$

When the operator under consideration is the density operator  $\hat{\rho}$ , then the associated phase-space functions

$$P_x(k,l;s) = \frac{1}{M} \text{Tr}[\hat{\rho} \hat{T}_x(k,l;s)] \quad (53)$$

can be regarded as  $s$ -parametrized quasidistributions in the discrete phase space. Equation (44) [with  $\hat{\rho}$  and  $P_x(k,l;s)$  in place of  $\hat{F}$  and  $f_x(k,l;s)$ , respectively] implies that knowledge of  $P_x(k,l;s)$  is equivalent to knowledge of the quantum state. The quasidistributions are normalized to unity,

$$\sum_{k,l} P_x(k,l;s) = 1 \quad (54)$$

[Eq. (49), with  $\hat{\rho}$  in place of  $\hat{F}$ , and  $\text{Tr}(\hat{\rho})=1$ ], and real. The latter follows from the fact that the phase-space functions of a Hermitian operator are always real [see Eq. (45) and recall that  $\hat{T}_x = \hat{T}_x^\dagger$ ]. According to Eq. (50), the quasidistributions can be used to calculate expectation values,

$$\text{Tr}(\hat{\rho}\hat{F}) = N \sum_{k,l} P_x(k,l;s) f_{-x}(k,l;-s). \quad (55)$$

The phase-space functions of the unity operator,  $u_x(k,l;s)$ , are independent of  $s$  and equal to

$$u_x(k,l;s) = \begin{cases} N^{-1}, & k,l \text{ integers} \\ 0, & k \text{ or } l \text{ half odd} \end{cases} \quad (56)$$

which follows directly from Eq. (45) and Eqs. (41) and (42). From Eqs. (56) and (50) we find that for even  $N$

$$\text{Tr}(\hat{F}) = \sum_{k,l \text{ integers}} f_x(k,l;s). \quad (57)$$

Equations (57) and (49) reveal that the sum of the phase-space function over points when at least one coordinate is half odd must be zero. The phase-space functions  $d_x(2n,-2m,k,l;s)$  associated with the displacement operator  $\hat{D}(2n,-2m)$  for  $N$  odd read as

$$\begin{aligned} d_x(2n,-2m,k,l;s) &= M^{-s-1} \tilde{W}_0^{-s}(xm,xn) \\ & \quad \times \exp\left[-i \frac{4\pi}{N} (km+ln)\right]. \end{aligned} \quad (58)$$

The convolutions of the phase-space functions  $f_x$  of an operator  $\hat{F}$  and the phase-space functions

$$P_1^{(0)}(m,n;s) = \frac{1}{M} \text{Tr}[\hat{\rho}_0 \hat{T}_1(m,n;s)] \quad (59)$$

of the reference-state density operator  $\hat{\rho}_0$  are again phase-space functions of  $\hat{F}$ :

$$\begin{aligned} \sum_{r,t} f_x(r,t;s_1) P_1^{(0)}[x(k-r), x(l-t); s_2] \\ = f_x(k,l; s_1 + s_2 - 1). \end{aligned} \quad (60)$$

(Using the function  $P_{-1}^{(0)}$  instead of  $P_1^{(0)}$  would correspond to changing the reference state  $\hat{\rho}_0$  by the parity transformed state  $\hat{P}\hat{\rho}_0\hat{P}$ .) When  $N$  is even and  $s$  an odd integer we find that the phase-space functions are zero for such points  $(k,l)$  where  $k$  or  $l$  is a half odd:

$$f_x(k,l;s) = 0, \quad s \text{ odd, } k \text{ or } l \text{ half odd} \quad (61)$$

which can be proved correct by using Eqs. (36) and (45) and recalling the properties (21) and (22) of the characteristic function.

The  $s$ -parametrized phase-space functions can also be expressed in terms of  $s$ -parametrized characteristic functions:

$$f_x(k,l;s) = \frac{1}{M} \sum_{m,n} \tilde{f}_x(m,n;s) \exp\left(-i \frac{4\pi}{N} (km + ln)\right), \quad (62)$$

where the  $s$ -parametrized characteristic functions  $\tilde{f}_x(m,n;s)$  are related to the characteristic function  $\tilde{f}(m,n)$  in Eq. (20) as

$$\tilde{f}_x(m,n;s) = M^{-s} \tilde{f}(m,n) \tilde{W}_0^{-s}(xm, xn). \quad (63)$$

We can see that for a reference state with even parity the two kinds of characteristic functions are equal to each other,  $\tilde{f}_1(m,n;s) = \tilde{f}_{-1}(m,n;s)$  [see Eq. (26)], and consequently the two kinds of phase-space functions are also equal to each other. The only important condition for the reference state is that its characteristic function cannot have zero values; otherwise it is impossible to define the phase-space functions with  $s > 0$ , see Eq. (36). Though we can formally define such functions with  $s < 0$ , we could not reconstruct from them the original operator  $\hat{F}$ . Let us mention two important examples of states which have characteristic functions with zero values. First, a state with completely uncertain (smeared) quantity  $U$  (or  $V$ ) would have zero value of  $\tilde{W}(1/2, 0)$  [or  $\tilde{W}(0, 1/2)$ ], see Eq. (35). Second, for even  $N$  any pure state  $|\psi\rangle$  whose expansion coefficients in the  $U$  basis  $\langle u_k | \psi \rangle$  are real (which could be, for example, some analog of a Gaussian state) we get, e.g.,  $\tilde{W}(1/2, N/4) = 0$ . Let us now consider special kinds of  $s$ -parametrized phase-space functions of the density operator.

#### D. Wigner function

We first consider the case when  $s=0$ . According to Eq. (36) the operators  $\hat{T}_1(k,l;0)$  and  $\hat{T}_{-1}(k,l;0)$  do not depend on the reference state and are equal to each other,  $\hat{T}(k,l) \equiv \hat{T}_1(k,l;0) = \hat{T}_{-1}(k,l;0)$ . The quasidistribution  $W(k,l) \equiv P_1(k,l;0) = P_{-1}(k,l;0)$  can be regarded as the discrete Wigner function,

$$W(k,l) = \frac{1}{N} \text{Tr}[\hat{\rho} \hat{T}(k,l)] \quad (64)$$

[see Eq. (53)]. In particular, for odd  $N$  the matrix elements of  $\hat{T}(k,l)$  in the  $U$  basis are given by

$$\langle u_r | \hat{T}(k,l) | u_s \rangle = \delta_{2k, r+s} \exp\left[i \frac{2\pi}{N} l(r-s)\right], \quad (65)$$

which is the Wootters definition [13] of the phase-space-point operator, so that for odd  $N$  Eq. (64) agrees with the Wootters definition of the discrete Wigner function. Applying Eq. (62) [together with Eq. (63)], Eq. (64) can be rewritten as

$$W(k,l) = \frac{1}{M} \sum_{m,n} \tilde{W}(m,n) \exp\left[-i \frac{4\pi}{N} (mk + nl)\right], \quad (66)$$

and we recognize the Leonhardt definition [22,23] of the discrete  $W$  function, used for both even and odd  $N$ .

The Wigner function exhibits a number of interesting properties. The trace of the product of two density operators can be calculated as the overlap

$$\text{Tr}(\hat{\rho} \hat{\rho}') = N \sum_{k,l} W(k,l) W'(k,l), \quad (67)$$

which follows directly from the general property (50). As was shown elsewhere [13,22,23], the marginal sums of the Wigner function are equal to the probabilities:

$$\sum_l W(k,l) = \langle u_k | \hat{\rho} | u_k \rangle, \quad (68)$$

$$\sum_k W(k,l) = \langle v_l | \hat{\rho} | v_l \rangle. \quad (69)$$

Further, marginal sums over special sets of phase-space points, the so-called ‘‘lines,’’ ‘‘broken lines,’’ etc. [13,23] are non-negative and are equal to probabilities of the system to be in some particular state.

In the case of continuous phase space it was shown [38,39] that the Wigner function is proportional to the mean value of the displaced parity operator. From Eqs. (25) and (65) we see that

$$\hat{T}(0,0) = \hat{P}. \quad (70)$$

Recalling Eq. (36) ( $s=0$ ), we verify that for odd  $N$  the operator  $\hat{T}(k,l)$  can be written as

$$\hat{T}(k,l) = \hat{D}(k,l) \hat{P} \hat{D}(-k, -l), \quad (71)$$

which means that also in the discrete phase space the  $W$  function can be defined (up to a constant) as the mean of the displaced parity operator. Note that for odd  $N$  the displaced parity operator  $\hat{D}(k,l) \hat{P} \hat{D}(-k, -l)$  represents all possible central symmetries of the discrete phase space, the center of symmetry being in the point with integer coordinates  $(k,l)$ . The situation is different in the case when  $N$  is even. First, the central symmetry with the center  $(k,l)$  ( $k,l$  integers) is

equivalent to the symmetries with centers  $(k, l+N/2)$ ,  $(k+N/2, l)$ , and  $(k+N/2, l+N/2)$  so that the function defined as the mean value of the displaced parity operator cannot contain sufficient information of the state  $\hat{\rho}$ . Second, there are such central symmetries, for which the center of symmetry is in the middle of two neighboring points with integer coordinates; as an example we can take the operator  $\hat{T}(1/2, 0) = \sum_r |u_r\rangle\langle u_{1-r}|$  (the summation runs over integers  $r$ ). We can consider such symmetries as parity operators “displaced by a half-odd value”; in contrast to the case of odd  $N$  they cannot be expressed in the form (71). This can serve as an explanation as to why, for even  $N$ , the phase space should consist of points with both integer and half-odd coordinates. However, for even  $N$  we can prove the relations

$$W(k, l+N/2) = (-1)^{2k} W(k, l), \quad (72)$$

$$W(k+N/2, l) = (-1)^{2l} W(k, l), \quad (73)$$

so that only  $N^2$  values are independent and just one-quarter of the phase space carries the whole information about the state  $\hat{\rho}$ . (Let us remember that the half-odd arguments are present also in the definition of the Wigner function in photon number and phase by Lukš and Peřinová [40]—here this fact reflects the symmetries of the enlarged Hilbert space.)

Let us also mention that the treatment of discrete phase space consisting from both integer and half-odd points can be used equally well for  $N$  odd. The formalism would be the same as for  $N$  even and we could use, e.g., Eqs. (72) and (73) for calculating the values in the points with half-odd coordinates. This may seem useless, but it gives us a unified look at the discrete phase-space formalism. It can also be used for showing the correspondence between discrete systems in the limit of large  $N$  and continuous systems. Let us sketch here the main idea, which will be presented in detail elsewhere. We can consider a state whose expansion coefficients in both  $U$  and  $V$  bases,  $\langle u_l | \psi \rangle$ ,  $\langle v_k | \psi \rangle$ , are non-negligible only for  $l$  and  $k$  inside some interval  $(-K, K)$ , where  $K \ll N$ . Let the expansion coefficients vary slowly with the change of the indices,  $|\langle u_{l+1} | \psi \rangle - \langle u_l | \psi \rangle| \ll 1$  and similarly for the  $V$  basis. Then in the phase space with both integer and half-odd coordinates, the  $W$  function of this state is represented by some structure in the vicinity of the origin, and by three similar, but highly oscillating structures shifted by  $(0, N/2)$ ,  $(N/2, 0)$ , and  $(N/2, N/2)$ . The sum over all the values of the central structure is approximately unity, whereas the sums of the values of the remaining structures are approximately zero. The central structure is a function slowly varying with the change of the indices by  $1/2$  and it can be treated as an analog of the continuous  $W$  function. On the other hand, if we would work only with integer coordinates, the discrete  $W$  function would be represented by the central structure, by two very similar structures shifted by  $(0, N/2)$  and  $(N/2, 0)$ , and by one more such structure with opposite sign shifted by  $(N/2, N/2)$ . The sum of values of the central structure is approximately  $1/2$ ; similarly for the other two “positive” structures, whereas the “negative” structure would give  $-1/2$ . It would therefore be difficult to interpret the central structure as the discrete analog of the continuous  $W$  function.

### E. Husimi function

Next let us consider the quasidistribution

$$Q(k, l) \equiv P_{-1}(k, l; -1) = \frac{1}{M} \text{Tr}[\hat{\rho} \hat{T}_{-1}(k, l; -1)]. \quad (74)$$

Using Eqs. (36) and (33) [with  $\hat{\rho}_0$  and  $\tilde{W}_0(m, n)$  in place of  $\hat{\rho}$  and  $\tilde{W}(m, n)$ , respectively], we derive that

$$\hat{T}_{-1}(k, l; -1) = \frac{M}{N} \hat{\rho}_0(k, l), \quad (75)$$

where  $\hat{\rho}_0(k, l)$  is the displaced reference-state density operator,

$$\hat{\rho}_0(k, l) \equiv \hat{D}(k, l) \hat{\rho}_0 \hat{D}(-k, -l). \quad (76)$$

Hence Eq. (74) can be rewritten as

$$Q(k, l) = \frac{1}{N} \text{Tr}[\hat{\rho} \hat{\rho}_0(k, l)], \quad (77)$$

which is the definition of the discrete  $Q$  function (or Husimi function) as given in [24, 25]. A similar definition was given recently also by Galetti and Marchioli [28]. As discussed in [24, 25], the  $Q$  function can be interpreted as proportional to the probability of the system to be in some state given by the displacement of the reference state—the so-called “quantum ruler.” (Note that a different normalization was used in [25].)

Let us mention some properties of the discrete  $Q$  function. From the relation (77) we can see that the  $Q$  function is always non-negative. Further, from Eq. (61) we see that for even  $N$  the  $Q$  function is nonzero only if both  $k$  and  $l$  are integers. Applying Eq. (60), we find that the  $Q$  function can be written as the correlation of the Wigner functions of the state studied,  $W(k, l)$ , and the reference state,  $W_0(k, l)$ ,

$$Q(k, l) = \sum_{r, t} W_0(r, t) W(r - k, t - l). \quad (78)$$

Since the reference state cannot be an eigenstate of the operator  $\hat{U}$  or  $\hat{V}$ , the discrete  $Q$  function can be interpreted as a quasiprobability distribution obtained from a “simultaneous” measurement of conjugated observables  $\hat{U}$  and  $\hat{V}$ . The measurement of the conjugated observables can formally be described as the “filtering” with a quantum ruler (see [24], and references therein). This filtering process is “responsible” for increase of quantum noise which results in “smearing” of the  $Q$  function compared to the Wigner function of the same state.

If we assume that the values of the  $Q$  function are obtained by some experiment, we can reconstruct the density operator as, on applying Eq. (44),

$$\begin{aligned} \hat{\rho} = & \frac{N}{M^3} \sum_{k, l} Q(k, l) \sum_{m, n} \tilde{W}_0^{-1}(m, n) \\ & \times \exp\left[-i \frac{4\pi}{N} (km + ln)\right] \hat{D}(-2n, 2m). \end{aligned} \quad (79)$$

We can see that even though the  $Q$  function can be defined in such cases when the characteristic function  $\widetilde{W}_0(m,n)$  of the reference state has zero values, it is then impossible to reconstruct from it the original density operator. This property is not dependent on our treatment based on the characteristic functions: one could, e.g., try to reconstruct the  $N^2-1$  independent elements of the density matrix solving a set of  $N^2-1$  linear equations whose right-hand sides are the  $N^2-1$  independent values of the measured  $Q$  function. But in this case the set is singular and we do not obtain the required solution.

### F. Glauber-Sudarshan function

The concept of quasidistributions also enables us to introduce the discrete  $P$  function. For this purpose we recall Eq. (44) [with  $\hat{\rho}$  and  $P_x(k,l;s)$  in place of  $\hat{F}$  and  $f_x(k,l;s)$ , respectively, and  $x=s=1$ ] and Eq. (75), which imply that

$$\hat{\rho} = \sum_{k,l} P(k,l) \hat{\rho}_0(k,l), \quad (80)$$

where

$$P(k,l) \equiv P_1(k,l;1) = \frac{1}{M} \text{Tr}[\hat{\rho} \hat{T}_1(k,l;1)] \quad (81)$$

[see Eq. (53)], and the displaced density operator of the reference state,  $\hat{\rho}_0$ , is given in Eq. (76). Equation (80) reveals that the density operator can be expanded in displaced reference-state density operators  $\hat{\rho}_0(k,l)$ , the expansion coefficients being given by the discrete phase-space function  $P(k,l)$ . This is quite similar to the infinite-dimensional Hilbert space, where the Glauber-Sudarshan  $P$  function  $P(\alpha)$  is introduced through the expansion of the density operator in coherent-state density operators  $|\alpha\rangle\langle\alpha|$ , i.e., displaced vacuum-state density operators, [3]

$$\hat{\rho} = \int d^2\alpha P(\alpha) |\alpha\rangle\langle\alpha|. \quad (82)$$

Thus we can consider the quasidistribution  $P(k,l)$  as a discrete analog of the Glauber-Sudarshan  $P$  function. In particular, the discrete  $P$  function of the displaced reference state  $\hat{\rho}_0(k',l')$  is equal to the Kronecker  $\delta$ ,

$$P(k,l) = \delta_{k,k'} \delta_{l,l'}, \quad (83)$$

which is analogous to the continuous case where the  $P$  function of the coherent state is the Dirac  $\delta$  function.

Similarly to the case of the  $Q$  function, for even  $N$  the  $P$  function is nonzero only at the points  $(k,l)$ , where both  $k$  and  $l$  are integers. From Eq. (60) we can see that the Wigner function can be calculated as the convolution of the  $P$  function with the Wigner function of the reference state,

$$W(k,l) = \sum_{r,t} P(r,t) W_0(k-r,l-t). \quad (84)$$

In our discussion we have not specified whether the quantum reference state described by the density operator is a pure state or a statistical mixture. Our formalism is equally

valid in both cases. Consequently, using the functions  $P(k,l)$  we can express the density operator of an arbitrary pure state as a sum of density operators of statistical mixtures. A similar situation was discussed in the case of infinite-dimensional Hilbert space [41], where generalized  $P$  and  $Q$  functions with thermal reference states were studied.

## III. APPLICATIONS

### A. Examples of discrete quasidistributions

To start our discussion we first have to specify the quantum reference state  $\hat{\rho}_0$ . For odd  $N$  we will consider this state to be the ground state of the Hamiltonian  $\hat{H}_0$ ,

$$\hat{H}_0 \equiv -\cos\hat{U} - \cos\hat{V}. \quad (85)$$

As shown in [36], such a state minimizes the Bandilla-Paul uncertainties for the quantities  $U$  and  $V$ . Increasing the dimension  $N$  of the Hilbert space to infinity, the Hamiltonian (85) can be approximated by the harmonic-oscillator Hamiltonian  $(1/2)\hat{U}^2 + (1/2)\hat{V}^2$  (for more discussion see [36]). Therefore the ground state of the Hamiltonian (85) can be considered as an analog of the vacuum state of the harmonic oscillator and it is a good candidate for the reference state in the formalism presented in this paper. In [25] it was chosen as the quantum ruler state in the definition of the discrete  $Q$  function. Because the ground state of the Hamiltonian (85) has the even parity, the two quasidistributions  $P_x(k,l;s)$  ( $x=\pm 1$ ) are mutually equal,  $P_1(k,l;s) = P_{-1}(k,l;s)$ .

In Fig. 1 we plot these quasidistributions for two special states in the odd- ( $N=11$ ) dimensional case. The first four diagrams present the situation when the studied state is equal to the reference state  $\hat{\rho}_0$ . In Fig. 1(a) the Wigner function  $P_x(k,l;0)$  of this state is presented. We see that in the vicinity of the origin of the phase space [i.e., around the point (0,0)] this Wigner function has a form of the discrete ‘‘hill’’ which resembles the Gaussian shape of the Wigner function of the vacuum state in the continuous case (in the limit of high  $N$  this similarity becomes closer). On the ‘‘edges’’ of the phase space the Wigner function is not equal to zero (as one would expect in the case of an exponentially ‘‘decaying’’ Gaussian function) but it exhibits rather complicated oscillatory behavior. Actually it consists of three similar structures as the central peak—two with the same sign and one with opposite sign. In spite of this behavior the marginal probability distribution functions  $W_u(k) = \sum_l W(k,l)$  and  $W_v(l) = \sum_k W(k,l)$  associated with probabilities to ‘‘observe’’ eigenstates of the operators  $\hat{U}$  and  $\hat{V}$ , respectively, are non-negative and normalized to unity.

Figure 1(b) shows the function  $P_x(k,l;1)$ ; this function represents the  $\delta$  Kronecker symbol which is the analog of the Dirac  $\delta$  function (i.e., the Glauber-Sudarshan  $P$  function of the vacuum state of the harmonic oscillator). Physically it means that for construction of the density matrix of the reference state we need only the reference state itself and not also its displaced versions.

Figure 1(c) presents the situation of  $s=-1$ , i.e., the discrete  $Q$  function. Properties of this function were studied in detail in [25]; here let us mention again the Gaussian-like



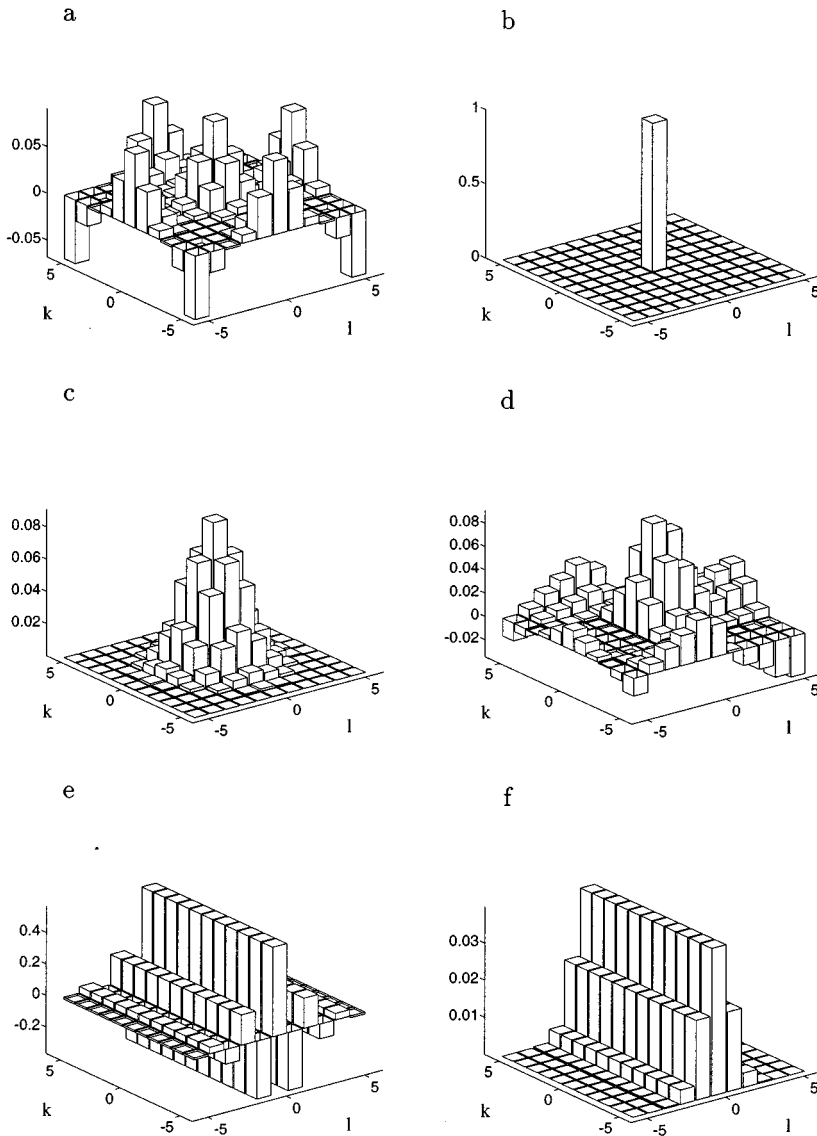


FIG. 1. Quasidistributions  $P_1(k,l;s) = P_{-1}(k,l;s)$  in the odd-dimensional case,  $N=11$ . The reference state is the ground state of Hamiltonian (85). (a)–(d) The studied state is equal to the reference state, (a)  $s=0$ , (b)  $s=1$ , (c)  $s=-1$ , (d)  $s=-0.5$ . (e), (f) The studied state is the “position state”  $|u_0\rangle$  (1), (e)  $s=1$ , (f)  $s=-1$ .

shape of the reference state; in comparison to the  $W$  function in Fig. 1(a) the peak is broader and the function takes only non-negative values.

In Fig. 1(d) we can see a transition from the  $W$  function to the  $Q$  function for the parameter  $s=-0.5$ . For this case the  $s$ -parametrized quasidistribution is not defined uniquely: we have to specify the definition of the  $s$ th power of a complex number for each phase-space point  $(k,l)$ . This can be done in a variety of ways; we must only satisfy the condition (38). In our case when the characteristic function  $\tilde{W}_0(m,n)$  is real (due to the even parity) this means that if in the point  $(m,n)$  the value of  $(-1)^s$  is defined as  $\exp(i\pi s)$ , then in the point  $(-m,-n)$  it must be defined as  $\exp(-i\pi s)$ . Once such a definition is adopted then the quasidistributions are defined for every real  $s$  and have all the important properties studied in the preceding section. However, as can be seen from this figure, due to the particular choice we lose the symmetry.

In Figs. 1(e) and 1(f) we show quasidistributions of the “position state” (or “line state”)  $|u_0\rangle$  (1); in Fig. 1(e) we see its  $P$  function ( $s=1$ ) which oscillates also to negative values, while Fig. 1(f) shows the smoothed shape of the  $Q$

function. (Let us note that the  $W$  function of such a state is nonzero only along the line  $l=0$  where it takes a constant value  $1/N$ .)

In Fig. 2 we present the case of  $N$  even; here  $N=6$ . As discussed in the previous text, the phase space consists of  $(2N)^2$  points—for each variable the values now run all integers and half odds between  $-3$  and  $2.5$ . For this situation we again have to specify the reference state. However, as follows from the discussion at the end of Sec. II C, in this case we cannot use the ground state of (85): such states only have real expansion coefficients in the  $U$  basis and therefore they have zero values of the characteristic function. To overcome this problem we constructed the reference state as the superposition of the ground state and the next excited state with even parity (the superposition factor of the excited state being  $0.1i$  times the factor of the ground state). However, the question of optimal choice of the reference states in the even  $N$  case remains open.

In Fig. 2(a) we show the  $W$  function ( $s=0$ ) of the reference state; again we can see the central peak in the point  $(0,0)$ ; however, its behavior around the “edges” of the phase space is more oscillatory than for the case of odd  $N$ .

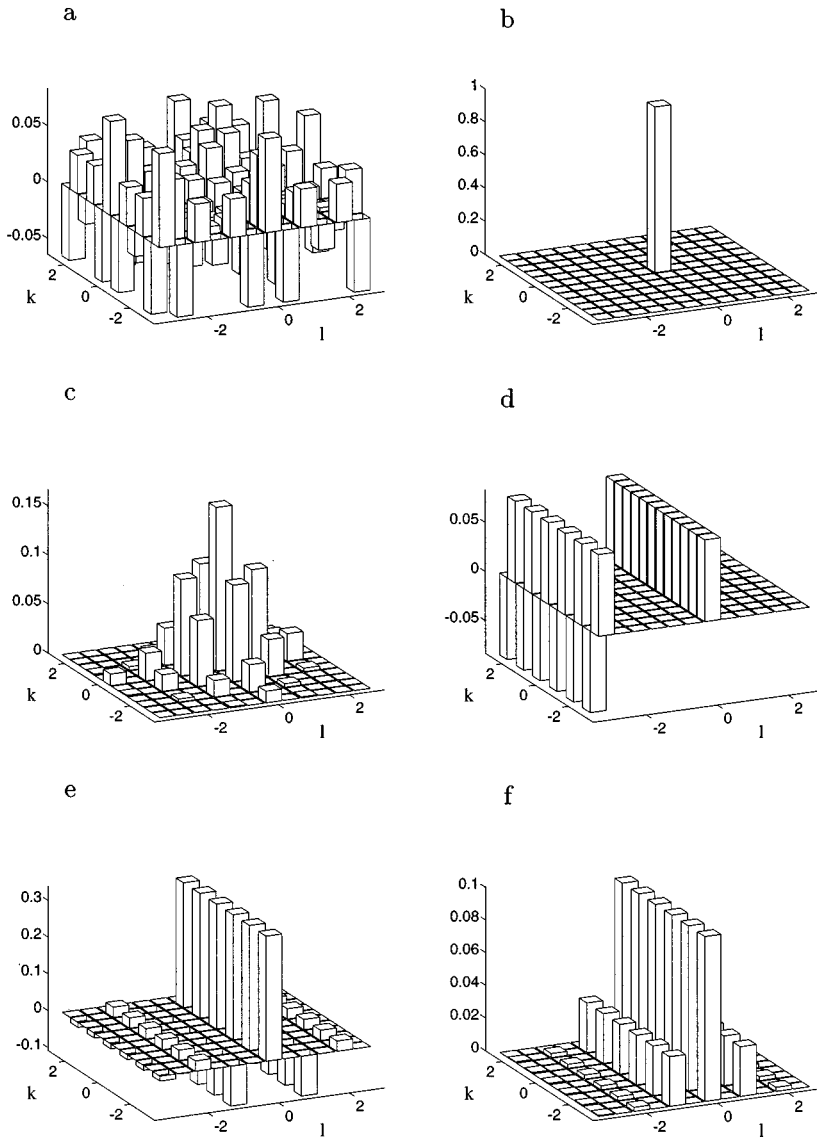


FIG. 2. Quasidistributions  $P_1(k, l; s) = P_{-1}(k, l; s)$  in the even-dimensional case,  $N=6$ . The reference state is a superposition of the ground state of Hamiltonian (85) and of the next excited even-parity state. (a)–(c) The studied state is equal to the reference state, (a)  $s=0$ , (b)  $s=1$ , (c)  $s=-1$ . (d)–(f) The studied state is  $|u_0\rangle$  (1), (d)  $s=0$ , (e)  $s=1$ , (f)  $s=-1$ .

Note the role of the negative values in points  $(k, l)$  with half odd  $k$  or  $l$ ; they ensure that the marginal sums over half-odd lines are zero. Figure 2(b) shows the  $P$  function ( $s=1$ ) of the reference state, again being equal to the Kronecker  $\delta$ , whereas Fig. 2(c) presents its  $Q$  function ( $s=-1$ ). Note that in the cases of  $s=\pm 1$  the corresponding quasidistributions are identically zero for any point with a half-odd coordinate and for any state—as follows from (61). In these figures we can see the broadening of the central peak from  $s=1$  over  $s=0$  to  $s=-1$ .

In Fig. 2(d) we can see the  $W$  function ( $s=0$ ) of the line state  $|u_0\rangle$ ; this Wigner function was discussed in detail by Leonhardt [23]. The marginal sum over values  $k$  for  $l=0$  gives us the probability 1 of the state to be found with this value of  $U$ ; on the other hand, such a marginal sum for  $l=-3$  over the oscillating terms gives zero. The negative values for  $l=-3$  also ensure the zero marginals over  $l$  for  $k$  half odd. Figure 2(e) shows the  $P$  function ( $s=1$ ) for this state and Fig. 2(f) its  $Q$  function ( $s=-1$ ). Whereas now the  $P$  function oscillates also to the negative values, the  $Q$  function is always non-negative.

Let us mention once more the question of choice of the reference state. Galetti and Marchioli [28] suggested a phase-space function based on coherent states defined as displacement of an eigenstate  $|\Phi_0\rangle$  of the Fourier transform which was calculated by Mehta [42]:

$$|\Phi_0\rangle \propto \sum_k \sum_{r=-\infty}^{\infty} \exp\left[-\frac{\pi}{N}(rN+k)^2\right] |u_k\rangle. \quad (86)$$

These states have properties similar to those of our ground states of (85) (which are also eigenstates of the Fourier transform), and their analytical expression contains the Gaussian form of the infinite-dimensional vacuum states. For  $N=2$  and  $N=3$  both definitions give the same result. Nevertheless, the problem of zero values of the characteristic function for  $N$  even is present also for the reference states defined as (86).

### B. A model of measurement of the discrete $Q$ function

There are many physical systems for which it is useful to work with the finite-dimensional Hilbert space formalism,

e.g., spin systems, atomic systems, quantum optical systems with a finite number of photons, etc. Here we would like to present a very simple and intuitive model of such a system and to show the principal possibility of the direct measurement of the discrete  $Q$  function.

Let us consider the system to be one monochromatic polarized photon, which can propagate in  $N$  parallel optical fibers—i.e., we are dealing with a multimode one-photon system. A pure state of this system is described by the state vector  $|\psi\rangle$ ,

$$|\psi\rangle = a_1|1\rangle_1|0\rangle_2\cdots|0\rangle_N + a_2|0\rangle_1|1\rangle_2\cdots|0\rangle_N + \cdots + a_N|0\rangle_1|0\rangle_2\cdots|1\rangle_N, \quad (87)$$

where the state vector  $|1\rangle_k$  means the one-photon Fock state in the  $k$ th mode. Thus the state vector  $|\psi\rangle$  can be written as the column vector

$$|\psi\rangle \leftrightarrow \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}, \quad (88)$$

and the squared amplitude  $|a_k|^2$  means the probability that the photon is in the  $k$ th fiber. We think that this system can correspond especially well to the Wootters discrete phase-space formalism, e.g., because these base states are without any *a priori* preferred state (in the spin systems there is always a state which is the “highest” and a state which is the “lowest”). Ordering the fibers to a ring-sectioned cable, we can immediately see the circular geometry of the system.

Let us assume that we can reproduce the studied state an arbitrary number of times (we have an infinite ensemble) and we want to measure it—i.e., to find its density matrix. For this purpose we can use energy conserving optical elements (representing unitary transformations) and photodetectors; see Fig. 3. Let us first show how various unitary transformations could be performed. One of the simplest transformations is represented by the displacement operator  $\hat{R}_u(n)$  (8). This would just correspond to rotating (or renumbering) the fibers—from the  $k$ th position to the position  $(k+n) \bmod N$ . The displacement  $\hat{R}_v(m)$  can also be easily performed—by connecting to the fibers phase elements, the phase shifter  $2\pi k/N$  to the  $k$ th fiber. Combining these two devices we can obtain any displacement  $\hat{D}(n,m)$  (11).

Any other unitary transformation can be constructed from these elements and a cascade of couplers, or equivalently beam splitters. (We can use the standard theory of beam splitters [43] to show that our one-photon multimode states transform in the same way as the multimode coherent states. For proof of the possibility to construct any unitary transformation using beam splitters see [44].)

Let us now show how a measurement of the discrete  $Q$  function can be performed. For this purpose we need one optical element representing the  $\hat{R}_u(n)$  rotations, one element representing the  $\hat{R}_v(m)$  phase shifts [thus having the displacement operator  $\hat{D}(n,m)$ ], one element for the reference state, and one detector. If the reference state is a pure

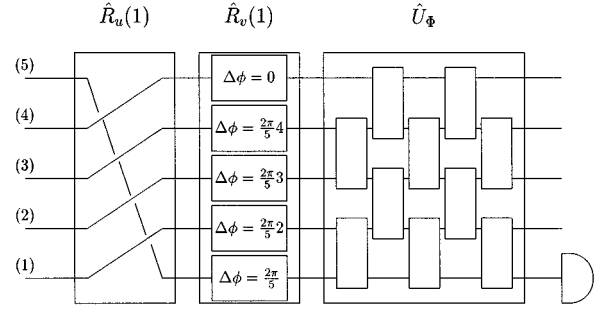


FIG. 3. Simple scheme for measuring the discrete  $Q$  function. The transformation  $\hat{R}_u(1)$  rotates the fibers,  $\hat{R}_v(1)$  represents phase shifters, and  $\hat{U}_\Phi$  describes a cascade of couplers. The detector is connected to the output of the first fiber.

state  $|\Phi\rangle \equiv |\Phi_1\rangle$ , the corresponding element can be described by the unitary operator  $\hat{U}_\Phi$ ,

$$\hat{U}_\Phi \equiv |u_1\rangle\langle\Phi_1| + \sum_{k=2}^N |u_k\rangle\langle\Phi_k|. \quad (89)$$

Here the vectors  $|\Phi_k\rangle$  form an orthonormal set. The important property of this device is that it transforms the reference state to the first fiber (i.e., to the state when the photon is with certainty in the first fiber). Let the detector be connected to this fiber; then it is described by the projection operator  $\hat{\Pi}_1 \equiv |u_1\rangle\langle u_1|$  and it measures the quantity  $\text{Tr}(\hat{\Pi}_1 \hat{\rho}_{\text{out}})$ , where  $\hat{\rho}_{\text{out}}$  is the output state of the system. If we connect our input state  $\hat{\rho}$  first to the displacement device and then the result to the reference-state device, the output state will be

$$\hat{\rho}_{\text{out}} = \hat{U}_\Phi \hat{D}(-n, -m) \hat{\rho} \hat{D}(n, m) \hat{U}_\Phi^\dagger, \quad (90)$$

and the probability of detecting the photon is

$$\begin{aligned} p(n, m) &= \text{Tr}(\hat{\Pi}_1 \hat{\rho}_{\text{out}}) \\ &= \langle u_1 | \hat{U}_\Phi \hat{D}(-n, -m) \hat{\rho} \hat{D}(n, m) \hat{U}_\Phi^\dagger | u_1 \rangle \\ &= \langle \Phi_1(n, m) | \hat{\rho} | \Phi_1(n, m) \rangle, \end{aligned} \quad (91)$$

where

$$|\Phi_1(n, m)\rangle \equiv \hat{D}(n, m) |\Phi_1\rangle \quad (92)$$

is the displaced reference state. We can see that up to the constant  $1/N$  this is just the discrete  $Q$  function of arguments  $(n, m)$ . Thus, the direct measurement of the  $Q$  functions is performed by changing the parameters  $n$  and  $m$  of the rotation operators  $\hat{R}_u(n)$  and  $\hat{R}_v(m)$  and by measuring the relative frequency of the detection of photons detected at the first fiber of the output.

As an example, let us now show a possible measurement of the  $Q$  function in this model for  $N=3$ . Let us choose the reference state as the ground state of the “Hamiltonian” (85). For  $N=3$  this operator has the matrix form (in the  $U$  as well as in the  $V$  representation, up to an unimportant additive constant)

$$\hat{H}_0 = \frac{1}{2} \begin{pmatrix} 1 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & 1 \end{pmatrix}. \quad (93)$$

The matrix of eigenvectors for this case can be written as

$$\hat{U}_\Phi = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_3 & a_4 & a_5 \\ a_5 & 0 & -a_5 \end{pmatrix}, \quad (94)$$

where

$$a_1 = [2(3+3^{1/2})]^{-1/2}, \quad (95)$$

$$a_2 = [(3+3^{1/2})/6]^{1/2}, \quad (96)$$

$$a_3 = -[2(3-3^{1/2})]^{-1/2}, \quad (97)$$

$$a_4 = [(3-3^{1/2})/6]^{1/2}, \quad (98)$$

$$a_5 = 2^{-1/2}. \quad (99)$$

The first row of this matrix corresponds to the reference state  $|\Phi\rangle$ . Because this matrix has one zero element, it can be constructed using only two couplers:

$$\hat{U}_\Phi = \hat{B}_1 \hat{B}_2, \quad (100)$$

where the coupler  $\hat{B}_1$  operates between the first and the second fibers,

$$\hat{B}_1 = \begin{pmatrix} \frac{1}{\sqrt{3+\sqrt{3}}} & \frac{1}{\sqrt{3-\sqrt{3}}} & 0 \\ \frac{1}{-\sqrt{3-\sqrt{3}}} & \frac{1}{\sqrt{3+\sqrt{3}}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (101)$$

while the coupler  $\hat{B}_2$  operates between the first and the third fibers,

$$\hat{B}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}. \quad (102)$$

Let us notice that for this simple model we can also easily avoid the noisy effect of the nonunity efficiency detection. Placing another detector to the remaining outputs we can disregard those measurements for which no detector detects the photon. Then the fraction of the number of ‘‘clicks’’ from the first detector to the total number of clicks is proportional (in the limit of infinite number of measurements) to the value of the  $Q$  function. Results of such measurements enable us to perform simple calculations to evaluate mean values of any operator  $\hat{F}$  in the state of interest: because we

measure the function  $P_{-1}(k,l;-1)$ , we can use the formula (55), with  $f_1(k,l;1)$  according to Eq. (45).

Let us mention that this measurement scheme enables also the tomographical reconstruction of the discrete  $W$  function, as suggested by Leonhardt [22]: for this we need to measure the projections of our state on the ‘‘discrete quadrature’’—the so-called line states, precessed line states, and broken line states (for their definition and more details see [22,23]). This would require a suitable change of the operator (89) so that it transforms a particular quadrature state into one fiber. Then the characteristic function of the probability distribution for such quadratures is proportional to the characteristic function of the discrete Wigner function.

#### IV. DISCUSSION AND CONCLUSION

In this work we were dealing with discrete phase-space formalism and we were trying to offer a consistent way for introducing  $s$ -parametrized phase-space functions to the quantum systems with a finite number of independent states. To summarize the main properties of these functions and to compare them to the situation in the infinite-dimensional Hilbert space we can draw the following conclusions.

(i) The  $s$ -parametrized phase-space functions are associated with a particular ‘‘reference’’ state. In the case of infinite-dimensional Hilbert space of a harmonic oscillator this reference state is usually implicitly assumed to be a vacuum state, even though sometimes also more general states are considered (e.g., for the squeezed  $Q$  function, squeezed  $P$  function, etc.).

(ii) The reference state can be chosen quite arbitrarily, the only condition is that its characteristic function must not have zero values. It is even possible to choose the reference state to be a statistical mixture.

(iii) The  $s$ -parametrized phase-space functions are equivalent descriptions of the operators. If the related operator is a density operator, we call these functions quasidistributions. From the quasidistribution we can calculate the density matrix and vice versa.

(iv) Properties of these functions (and of the phase spaces) are different for the case of odd and even  $N$  dimension of the Hilbert space. For  $N$  odd the phase space consists of  $N \times N$  integer numbers, whereas for  $N$  even it consists of  $2N \times 2N$  numbers—integers and half odds. This follows from the different symmetries of the phase spaces. However, we can consistently use the formalism of half-odd phase-space points also in the case of  $N$  odd; this provides no new information, but it may be useful for observing the correspondence between the continuous phase-space functions and their discrete counterparts in the limit of large  $N$ .

(v) Generally there are two kinds of phase-space functions for each  $s$ ,  $P_1(k,l;s)$  and  $P_{-1}(k,l;s)$ ; the exceptions are for  $s=0$  and for the case when the reference state has even parity, in which cases these two functions are equal. To our knowledge this situation was not discussed for the continuous case, where only even-parity reference states were considered.

(vi) With a properly chosen reference state the  $s$ -parametrized phase-space functions are uniquely defined for every integer  $s$ . For  $s$  real (noninteger) an ambiguity stems from a nonunique definition of real powers of complex

numbers. Specifying these powers we can define discrete phase-space functions for arbitrary real  $s$ —even though such a choice is artificial and there will always be other alternative definitions. This problem has not occurred in the continuous case because (to our knowledge) only Gaussian reference states were considered, which have real positive characteristic functions.

(vii) The most important cases of these quasidistributions are for  $s=0$ ,  $P_1(k,l;0)=P_{-1}(k,l;0)=W(k,l)$ , the Wigner function, for  $s=-1$ ,  $P_{-1}(k,l;-1)=Q(k,l)$  the  $Q$  function, and for  $s=1$ ,  $P_1(k,l;1)=P(k,l)$  the  $P$  function. The meaning of the  $Q$  function is that it is proportional to probabilities of the state to be in a displaced reference state, whereas the  $P$  function enables us to write the density matrix of the studied state as a sum of displaced reference-state density matrices. Contrary to the  $W$  function in the case of  $N$  even, both  $P$  and  $Q$  functions (and any other phase-space function with  $s$  odd) are nonzero only in points with integer coordinates.

(viii) Similarly to the continuous case, the  $s$ -parametrized quasidistributions enable us to simply calculate squared scalar products of two states and/or mean values of operators.

(ix) A convolution of a phase-space function  $f_1$  (or correlation of  $f_{-1}$ ) of some studied operator  $\hat{F}$  with the function

$P_1$  of the reference state is a phase-space function of the operator  $\hat{F}$ .

We also offer a very simple measurement scheme which shows a connection of the discrete  $Q$  function to an experiment. It is based on the possibility to construct physical realization of the discrete displacement operators and of the operator which transforms the reference state to the detector input. Because the reference state cannot be an eigenstate of either of the phase-space-variable operators ( $\hat{U}$  or  $\hat{V}$ ), we can look at such measurements as smeared simultaneous measurements of these quantities. This scheme also enables the discrete tomographical reconstruction of the  $W$  function.

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