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# Operational phase distributions via displaced squeezed states

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**Abstract.** We introduce an 'effective' phase state with which high-sensitivity phase-shift measurements can be performed. These states can be used for high-precision operational measurements of phase distributions of states which in phase space are localized in regions whose angular width is much larger than $\pi$ We propose a method by means of which the corresponding operational phase distribution can be measured.

## 1. Introduction

In interferometers with coherent-state inputs the accuracy to which a phase shift can be measured is proportional to $1/\langle N\rangle^{1/2}$, where $\langle N\rangle$ is the total number of photons which enter the beam splitter. The accuracy can be improved to $1/\langle N\rangle$ if different input states are used. In particular, if two single-mode squeezed states are used as inputs in the two ports of an interferometer, then this accuracy can be achieved if homodyne measurements are made at the output [1]. This accuracy can also be achieved in an interferometer in which photon-counting measurements are performed at the output if more 'exotic' input states are used [2].

The work on the detection of phase shifts has led to a renewed interest in the quantum-mechanical definition of the phase of a field mode (for a recent review, see [3]). One way of approaching this problem is through a distribution function for the phase of a field mode. Quantum phase distributions go back to the work of London [4] in 1926. More recent approaches which are closely related to that of London are contained in the work of Pegg and Barnett [5] and Shapiro and Shepard [6]. Pegg and Barnett [5] arrive at a phase distribution by defining a hermitian phase operator and its eigenstates on a finite-dimensional Hilbert space and finding the probability that a field state in this space is in one of the phase eigenstates. A phase distribution for a general field state is defined by letting the number of dimensions go to infinity. Instead of using a formalism of finite-dimensional Hilbert space and the limiting procedure, Shapiro and Shepard [6] work with a positive-operator-valued measure in the full infinite-dimensional Hilbert space. All three of these analyses lead to the same phase distribution function.

Different quantum-phase distributions can be found if one starts with quasi-probability distribution functions for quantum states, such as the Wigner or

Husimi functions. These are functions of the real and imaginary parts of the complex field amplitude and a phase distribution function is derived from them by expressing their arguments in terms of polar coordinates and integrating out the radial coordinate [3, 7, 8]. In the case of the Husimi or $Q$ function the procedure yields a distribution function which is positive. However, in the case of the Wigner function, as shown by Garraway and Knight [9], this is not true. Thus the integrated Wigner phase distribution can be regarded only as a quasidistribution function.

Yet another method of defining a phase distribution function is operationally. That is, one specifies a procedure for the measuring the phase and determines what kind of a distribution function is produced. There are at present two such schemes both of which are based on homodyne measurements. One was developed by Vogel and Schleich [10] and the other by Noh *et al.* [11]. Here we shall propose a third operational scheme based on displaced squeezed states which can supplement the information provided by the other two.

Before considering the distribution function developed by Vogel and Schleich, which is closely related to that discussed in this paper, let us ask what properties we would like a quantum distribution to have. This will help us to determine what properties of our proposed distribution we should analyse. One of the main reasons for looking at quantum phase distributions is that we would like them to tell us something about phase shift measurements. In particular, if we are given the phase distribution for a particular quantum state, we would like to determine from the distribution function whether this state would be useful in making phase shift measurements, and what the accuracy of these measurements would be. The distributions which are most closely related to phase shift measurements are the London or Pegg–Barnett distributions, and the integrated $Q$ function [6, 12]. The former is better for the analysis of phase-shift sensitivity because it resolves the phase properties of a state better; the integrated $Q$ function is a broader noisier distribution. One would also like a phase distribution to be, in some sense, measurable. This latter requirement is automatically satisfied for an operationally defined distribution function. Therefore we shall want to consider how our proposed phase distribution is related to phase shift measurements and how it can be measured.

Vogel and Schleich [10] have proposed an operational phase distribution $P_{\mathrm{VS}}(\theta)$ which is defined as

$$P_{\mathrm{VS}}(\theta) = \mathcal{N}\, |{}_{\mathrm{VS}}\langle \Phi(\theta) | \Psi\rangle|^2, \tag{1}$$

where $\mathcal{N}$ is a normalization constant, $|\Psi\rangle$ is the state for which a phase distribution is going to be measured and $|\Phi(\theta)\rangle_{\mathrm{VS}}$ is the 'phase' state as introduced by Vogel and Schleich. This state is equal to an eigenstate $|x_{\theta+\pi/2}\rangle$ with eigenvalue 0 of the rotated quadrature operator $\hat{x}_{\theta+\pi/2}$ (see below). In phase space the Vogel–Schleich phase state is represented by a line which runs through the origin and which makes an angle $\theta$ with the $q$ axis. On the other hand, as was noted by Vogel and Schleich, a 'true' phase state corresponding to the angle $\theta$ should be represented as a ray rather than a line, starting at the origin of the phase space and making an angle $\theta$ with the positive $q$ axis (for a given reference angle $\theta_0 = 0$). Consequently, the Vogel–Schleich approach should provide reasonable phase distributions for states which are localized in phase space in regions of angular width less than $\pi$, and it definitely cannot be applied to states such as a superposition of coherent states of the form $(|\alpha\rangle + \exp(i\phi)|-\alpha\rangle)$. In principle, the Vogel–Schleich phase states $|\Phi(\theta)\rangle_{VS} \equiv |x_\theta\rangle$ are difficult to realize because they correspond to single-mode squeezed states with infinite squeezing. However, the square of the inner product appearing in equation (1) can be measured to high accuracy using homodyne measurements.

In the present paper we introduce an 'effective' phase state which shares one of the attractive features of the Vogel–Schleich formalism, a high sensitivity to phase shifts. In addition these states can be used for measurements of phase distributions of states which in phase space are localized in regions whose angular width is much larger than $\pi$.

Our paper is organized as follows. In section 2 we briefly review the Vogel–Schleich formalism and we introduce new operational phase states. In section 3 we propose a method by means of which the corresponding operational phase distribution can be measured. In section 4 we analyse properties of the operational phase distribution. We find that it can be used in determining the utility of a state for phase-shift measurements and that in a certain limit it can be used to find the Pegg–Barnett or London phase distribution of a state.

## 2. Operational phase states

Let us consider a single-mode light field associated with the annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$ respectively, such that $[\hat{a}, \hat{a}^\dagger] = 1$. These operators can be expressed as a complex linear combination of quadrature operators $\hat{q}$ and $\hat{p}$, where $[\hat{q}, \hat{p}] = i\hbar$ (in what follows we use units such that $\hbar = 1$):

$$\hat{a} = \frac{1}{2^{1/2}}(\hat{q} + i\hat{p}), \qquad \hat{a}^\dagger = \frac{1}{2^{1/2}}(\hat{q} - i\hat{p}). \tag{2}$$

The operators $\hat{q}$ and $\hat{p}$ represent special realizations of a more general rotated quadrature operator $\hat{x}_\theta$:

$$\hat{x}_\theta \equiv \frac{1}{2^{1/2}}[\hat{a}\exp(-i\theta) + \hat{a}^\dagger \exp(i\theta)]. \tag{3}$$

From equations (2) and (3) it follows that $\hat{q} = \hat{x}_{\theta=0}$ and $\hat{p} = \hat{x}_{\theta=\pi/2}$. Generally, two operators $\hat{x}_\theta$ and $\hat{x}_{\theta+\pi/2}$ which are mutually 'shifted' by $\pi/2$ are conjugate operators which have the property that

$$[\hat{x}_\theta, \hat{x}_{\theta+\pi/2}] = i. \tag{4}$$

The eigenstates $|x, \theta\rangle$ of the rotated quadrature operator $\hat{x}_\theta$, that is

$$\hat{x}_\theta |x, \theta\rangle = x|x, \theta\rangle \tag{5}$$

are, in general, characterized by two parameters. The first is an eigenvalue of the operator $\hat{x}_\theta$ (for instance, the mean amplitude of an electromagnetic field). The second parameter is the phase $\theta$. The eigenstates $|x, \theta\rangle$ of the operator $\hat{x}_\theta$ have an infinite energy, that is the mean photon number $\hat{N} = \hat{a}^\dagger \hat{a}$ in these states diverges. Therefore it is very useful to introduce 'regularized' states which in some limit tend to $|x, \theta\rangle$. To be more specific, let us consider the eigenstate $|q\rangle \equiv |q, \theta = 0\rangle$

of the position operator $\hat{q}$. This state can be represented as the displaced squeezed state $|q, -r\rangle$ [13] in the limit of infinite squeezing, that is

$$|q\rangle = \lim_{r\to\infty} [\eta(r)|q, r\rangle], \tag{6}$$

where $\eta(r)$ is an appropriate normalization factor (for $\delta$-function normalization $\eta(r) = (\exp r)/2\pi^{1/2}$). The displaced squeezed state $|q, r\rangle$ is defined by the action of the displacement operator $\hat{D}(q, p)$ [14]:

$$\hat{D}(q, p) = \exp [\mathrm{i}(p\hat{q} - q\hat{p})], \tag{7}$$

and the squeezing operator $\hat{S}(r)$ (with real squeezing parameter $r$) [15]

$$\hat{S}(r) = \exp\left[\frac{\mathrm{i}r}{2}(\hat{q}\hat{p} + \hat{p}\hat{q})\right] \tag{8}$$

on the vacuum state $|0\rangle$, so that

$$|q, r\rangle \equiv \hat{D}(q, 0)\hat{S}(r)|0\rangle. \tag{9}$$

The action of the position operator $\hat{q}$ on the state $|q, r\rangle$ is

$$\hat{q}|q, r\rangle = q|q, r\rangle + \exp(-r)\frac{1}{2^{1/2}}\hat{D}(q, 0)\hat{S}(r)|1\rangle, \tag{10}$$

where $|1\rangle$ is a Fock state with one photon. For the mean value of the operator $\hat{q}^n$ in the state $|q, r\rangle$ we find that

$$\langle q, r|\hat{q}^n|q, r\rangle = \sum_{m=0}^{[n/2]} \binom{n}{2m} q^{n-2m}(2m-1)!!\left(\frac{\exp(-2r)}{2}\right)^m, \tag{11}$$

where $[x]$ denotes the largest integer smaller than $x$. From equations (10) and (11) it is clear that in the limit of infinite squeezing (i.e. in the limit $r \to \infty$ when $\langle q, r|\hat{q}^n|q, r\rangle \to q^n$) the state $|q, r\rangle$ is the eigenstate of the operator $\hat{q}$. We also note that the mean photon number in the state $|q, r\rangle$ is

$$\langle q, r|\hat{N}|q, r\rangle = \frac{q^2}{2} + \sinh^2 r, \tag{12}$$

from which it is clear that in the limit $r \to \infty$ the energy of the eigenstate of the position operator diverges exponentially.

### 2.1. *Vogel–Schleich phase distribution*

Following Vogel and Schleich [10] one can identify the phase state $|\Phi(\theta)\rangle_{\mathrm{VS}}$ with the eigenstate of the operator $\hat{x}_{\theta+\pi/2}$ with the zero eigenvalue (here the reference angle, i.e. the phase window, is chosen in such way that $\theta = 0$ corresponds to the direction of the positive $q$ axis in phase space). From equation (9) we see that the Vogel–Schleich phase state $|\Phi(\theta)\rangle_{\mathrm{VS}}$ is in some sense the limit of a squeezed vacuum state as the squeezing goes to infinity. In particular, the Vogel–Schleich phase distribution of a particular state $|\Psi\rangle$ can now be defined in a straightforward quantum-mechanical way as an overlap of the state $|\Psi\rangle$ with the state $|\Phi(\theta)\rangle_{\mathrm{VS}}$, that is

$$P_{\mathrm{VS}}(\theta) = \mathcal{N}\,|\langle\Psi|\Phi(\theta)\rangle_{\mathrm{VS}}|^2 = \mathcal{N} \lim_{r\to\infty} |\langle\Psi|\hat{S}(r\exp(2\mathrm{i}\theta + \mathrm{i}\pi))|0\rangle|^2, \tag{13}$$

where the normalization constant $\mathcal{N}$ is defined in such way that

$$\int_{-\pi/2}^{\pi/2} P_{\mathrm{VS}}(\theta)\,\mathrm{d}\theta = 1. \tag{14}$$

We note that as a consequence of the definition (13) the phase distribution $P_{\mathrm{VS}}(\theta)$ is $\pi$-periodic rather than $2\pi$-periodic which one would expect for a 'true' phase distribution, such as the London [4] or Pegg–Barnett [5] phase distribution.

## 2.2. *Phase distribution via displaced squeezed states*

To overcome the problem of $\pi$ periodicity of the Vogel–Schleich phase distribution we utilize a displaced squeezed state $|\Phi(\theta)\rangle$:

$$|\Phi(\theta)\rangle \equiv \hat{D}(s\exp(\mathrm{i}\theta))\hat{S}(r\exp(2\mathrm{i}\theta))|0\rangle = \hat{U}(\theta)\hat{D}(s)\hat{S}(r\exp(\mathrm{i}\pi))|0\rangle \tag{15 a}$$

or

$$|\Phi(\theta)\rangle \equiv \hat{U}(\theta)|\Phi\rangle, \qquad |\Phi\rangle \equiv \hat{D}(s)\hat{S}(r)|0\rangle \tag{15 b}$$

as an effective phase state. The phase shift operator $\hat{U}(\theta)$ in equations (15) is defined as

$$\hat{U}(\theta) = \exp(\mathrm{i}\hat{N}\theta), \tag{16}$$

and the displacement operator $\hat{D}(\alpha)$ (see equation (7)) is given in terms of photon creation and annihilation operators by $\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$, where $\mathrm{Re}\,\alpha = q/2^{1/2}$ and $\mathrm{Im}\,\alpha = p/2^{1/2}$. In equations (15), $\alpha$ is taken to be equal to $s$ which is assumed to be real and greater than or equal to zero. In particular, for $s = 0$ we obtain from equations (15) the expression for the squeezed vacuum state:

$$\lim_{s\to 0} |\Phi(\theta)\rangle = \hat{S}(r\exp(2\mathrm{i}\theta))|0\rangle, \tag{17 a}$$

which for $\theta = 0$ can be expressed in the Fock basis as

$$\lim_{s\to 0} |\Phi(\theta=0)\rangle = \exp\left(\frac{r}{2}[(\hat{a}^\dagger)^2 - \hat{a}^2]\right)|0\rangle = \frac{1}{(\cosh r)^{1/2}} \sum_{n=0}^{\infty} \frac{[(2n)!]^{1/2}}{2^n n!}(\tanh r)^n|2n\rangle. \tag{17 b}$$

The variances of the operators $\hat{q}$ and $\hat{p}$ in the squeezed vacuum (17) are

$$\langle(\Delta\hat{q})^2\rangle = \tfrac{1}{2}\exp(2r), \qquad \langle(\Delta\hat{p})^2\rangle = \tfrac{1}{2}\exp(-2r), \tag{18}$$

which means that phase fluctuations are reduced in the direction perpendicular to the $q$ axis.

Using the above definition of effective phase states (15) we can introduce the corrsponding phase distribution $P(\theta; s, r)$ as

$$P(\theta; s, r) = \mathcal{N}|\langle\Psi|\Phi(\theta)\rangle|^2 = \mathcal{N}|\langle\Psi|\hat{D}(s\exp(\mathrm{i}\theta))\hat{S}(r\exp(2\mathrm{i}\theta + \mathrm{i}\pi))|0\rangle|^2. \tag{19}$$

The normalization constant $\mathcal{N}$ is such that

$$\int_{-\pi}^{\pi} P(\theta; s, r)\,\mathrm{d}\theta = 1. \tag{20}$$

From the definition (15) of the effective phase state $|\Phi(\theta)\rangle$ we see that this state is displaced and stretched in the direction $\theta$ which significantly reduces the contribution of the phase fluctuations of the phase states themselves to the distribution $P(\theta; s, r)$. Moreover, owing to the displacement the $2\pi$-periodicity of the phase distribution is (partially) restored. If $s=0$ (i.e. when there is no displacement), then $|\Phi(\theta)\rangle$ given by equations (15) serves as a prototype of the Vogel–Schleich phase state.

### 2.3. *Example A*

To illustrate the ideas presented above let us analyse the phase distribution of a coherent state $|\xi\rangle = \hat{D}(\xi)|0\rangle$ with a real displacement parameter $\xi = q/2^{1/2}$. After some algebra we find that

$$|\langle\xi|\Phi(\theta)\rangle|^2 = \frac{1}{\cosh r}\exp\left[-s^2\exp(-2r)\tanh r - \xi^2 - s^2\exp(-2r)\right] \times \exp\left[\xi^2\tanh r\cos(2\theta) + \frac{2}{\cosh r}\xi s\exp(-r)\cos\theta\right], \tag{21}$$

from which the phase distribution (19) can be obtained. From (21) it follows that the mean angle $\bar{\theta} = \int_{-\pi}^{\pi} P(\theta; s, r)\theta\,d\theta$ is equal to zero as it has to be for real $\xi$. We note that, in the limit $r \to 0$, equation (21) reduces to

$$\lim_{r\to 0}|\langle\xi|\Phi(\theta)\rangle|^2 = \exp\left[-|\xi - s\exp(i\theta)|^2\right], \tag{22}$$

which reflects the fact that the state $|\Phi(\theta)\rangle$ in the limit $r \to 0$ is equal to a coherent state $|s\exp(i\theta)\rangle$. The normalized phase distribution (19) corresponding to this situation, that is when the phase state is approximated by a coherent state $|s\exp(i\theta)\rangle$, is $2\pi$-periodic.

If on the other hand we assume that $s \to 0$, then equation (21) reduces to

$$\lim_{s\to 0}|\langle\xi|\Phi(\theta)\rangle|^2 = \frac{1}{\cosh r}\exp\{-\xi^2[1 - \tanh r\cos(2\theta)]\}. \tag{23}$$

Using this expression we obtain a phase distribution which corresponds to a situation when the 'true' phase state is approximated by a squeezed vacuum $\hat{S}(r\exp(2i\theta))|0\rangle$. This distribution is just $\pi$-periodic and in the limit $r \to \infty$ is equal to the Vogel–Schleich phase distribution of a coherent state $|\xi\rangle$.

Let us now consider the mean photon number of the phase state $|\Phi(\theta)\rangle$ to be fixed and equal to the mean photon number of the measured coherent state $|\xi\rangle$, that is $\bar{n} = \sinh^2 r + s^2 = \xi^2$. In figure 1 (*a*) we plot the variance $(\Delta\theta)^2$ of the distribution corresponding to equation (21). This variance is (we remind ourselves that $\bar{\theta} = 0$)

$$(\Delta\theta)^2 = \int_{-\pi}^{\pi} d\theta\,\theta^2|\langle\xi|\Phi(\theta)\rangle|^2 \bigg/ \int_{-\pi}^{\pi} d\theta\,|\langle\xi|\Phi(\theta)\rangle|^2. \tag{24}$$

The variance $(\Delta\theta)^2$ is plotted as a function of the parameter $s$ which range from 0 to $\bar{n}^{1/2}$ in such a way that

$$\bar{n} = \sinh^2 r + s^2 = \xi^2 = \text{constant}. \tag{25}$$

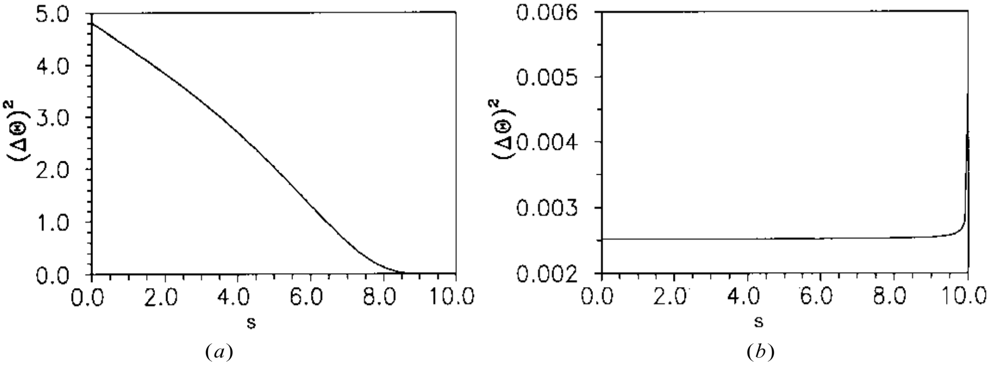


Figure 1. (*a*) Plot of the variance $(\Delta\theta)^2$ corresponding to the phase distribution normalized on the interval $[-\pi, \pi]$ given by equation (24) as a function of $s$. (*b*) The phase distribution considered to be normalized on the interval $[-\pi/2, \pi/2]$ (see equation (26)).

This means that for $s = 0$ the phase state is approximated by the squeezed vacuum characterized by the squeezing parameter $r$ such that $\bar{n} = \sinh^2 r$ (see equation (22)). On the other hand, if $s^2 = \bar{n}$, then the phase state is approximated by a coherent state (see equation (22)). From figure 1 (*a*) we see that in the case of the squeezed vacuum used as a phase state ($s = 0$) the fluctuations $(\Delta\theta)^2$ in the phase distribution are very large owing to the $\pi$-periodicity of this distribution. On the other hand for small values of $r$ the fluctuations $(\Delta\theta)^2$ are approximately equal to those of the case when the true phase state is approximated by a coherent state with the amplitude $s \approx \xi$. For comparison purposes we plot in figure 1 (*b*) the variance $(\Delta\theta)^2$ in the case when the phase distribution is normalized on the interval $[-\pi/2, \pi/2]$:

$$(\Delta\theta)^2 = \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta\, \theta^2 |\langle \xi | \Phi(\theta) \rangle|^2 \Big/ \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta\, |\langle \xi | \Phi(\theta) \rangle|^2. \tag{26}$$

In the limit $s = 0$ this is a prototype of the variance obtained of the phase distribution of a coherent state obtained in the framework of the Vogel–Schleich formalism. From figure 1 (*b*) we see that in this case fluctuations $(\Delta\theta)^2$ are smallest for a given value of $\bar{n}$ when $s = 0$, but what is important is that, even for a significant displacement $s$ (which is performed at the expense of a reduction in the squeezing; see condition (25)), fluctuations in phase remain constant and approximately at the level of the squeezed vacuum case. In other words, a very modest squeezing of a coherent state, which plays a role of the phase state, improves significantly the precision with which the phase can be determined. This means that, if we plan to implement the Vogel–Schleich scheme in a real experiment, that is instead of eigenstates of the rotated quadrature operator we consider (highly) squeezed states with finite mean photon number, the best accuracy that we can obtain is the same as in the case when we measure the phase distribution with slightly squeezed coherent states of the same intensity $\bar{n}$. Taking into account that it is much easier to displace than to squeeze, the advantage of the effective phase state given by equations (15) is transparent.

### 2.4. *Example B*

There are states for which the Vogel–Schleich phase distributions does not work even approximately. To illustrate this let us consider the so-called odd coherent state [16] which is defined as a particular superposition of two coherent states $|\xi\rangle$ and $|-\xi\rangle$:

$$|\xi\rangle_{\mathrm{odd}} = \{2[1 - \exp(-2\xi^2)]\}^{-1/2}(|\xi\rangle - |-\xi\rangle), \tag{27}$$

where we assume $\xi$ to be real. It is easy to see that the Vogel–Schleich phase states $|\Phi(\theta)\rangle_{\mathrm{VS}}$ given by equation (13) are orthogonal to the odd coherent state, that is ${}_{\mathrm{VS}}\langle\Phi(\theta)|\xi\rangle_{\mathrm{odd}} = 0$, which means that $|\Phi(\theta)\rangle_{\mathrm{VS}}$ cannot be defined. On the other hand, if we approximate phase states by displaced squeezed states, then we find

$$\begin{aligned}|{}_{\mathrm{odd}}\langle\xi|\Phi(\theta)\rangle|^2 = {} & \frac{1}{\cosh r\,[1 - \exp(-2\xi^2)]} \\ & \times \exp[-s^2 \exp(-2r)\tanh r - \xi^2 - s^2\exp(-2r)] \\ & \times \exp[\xi^2 \tanh r \cos(2\theta)]\left[\cosh\left(\frac{2}{\cosh r}\xi s \exp(-r)\cos\theta\right)\right. \\ & \left. - \cos\left(\frac{2}{\cosh r}\xi s\exp(-r)\sin\theta\right)\right].\end{aligned} \tag{28}$$

From equation (28) it directly follows that

$$\lim_{s\to 0} |{}_{\mathrm{odd}}\langle\xi|\Phi(\theta)\rangle|^2 = |\langle 0|\hat{S}^\dagger(r\exp(2\mathrm{i}\theta))|\xi\rangle_{\mathrm{odd}}|^2 = 0, \tag{29}$$

which illustrates the fact that the squeezed vacuum is orthogonal to the odd coherent state. This result can be easily understood if we remind ourselves that the squeezed vacuum is, in the Fock basis, represented as a superposition of even Fock states $|2n\rangle$ (see equation (17 *b*)] while the odd coherent state (27) is represented as a superposition of only odd Fock states $|2n+1\rangle$.

In figure 2 we plot the phase distribution $P(\theta; s, r)$ (see equation (19)) which is evaluated with the help of equation (28). We assume that the mean photon number of the 'phase' state $|\Phi(\theta)\rangle$ is equal to the mean photon number of the measured odd coherent state, that is

$$\bar{n} = \sinh^2 r + s^2 = \frac{\xi^2[1 + \exp(-2\xi^2)]}{1 - \exp(-2\xi^2)} = \text{constant}, \tag{30}$$

which is chosen to be equal to 100. In figure 2 (*a*) we plot $P(\theta; s, r)$ in the case when $r = 0$ and $s = 10$, that is the 'true' phase state is approximated by a coherent state $|s\exp(\mathrm{i}\theta)\rangle$. The two-peak structure of the phase distribution corresponding to the odd coherent state is transparent. This structure is even better pronounced if we suppose the state $|\Phi(\theta)\rangle$ to be slightly squeezed. In figure 2 (*b*) we consider the squeezing parameter $r$ to be such that $\sinh^2 r = s^2 = 50$, while in figure 2 (*c*) we assume that $\sinh^2 r = 36$ and $s^2 = 64$. From these figures we again see that even a very small amount of squeezing of the coherent states which serve as an approximation of the true phase states significantly improves the sensitivity with which the phase distribution can be measured.

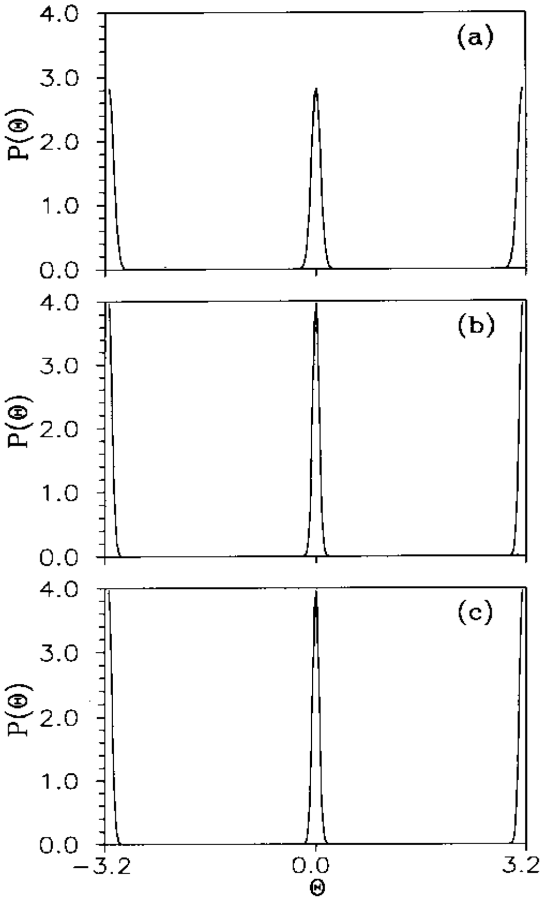


Figure 2. The phase distribution $P(\theta; s, r)$ of the odd coherent state (see equation (28)), where the mean photon numbers in the odd coherent state $|\xi\rangle_{\text{odd}}$ and the phase state $|\Phi(\theta)\rangle$ are considered to be equal to 100: (*a*) the phase state approximated by a coherent state with $s^2 = 100$; (*b*) $s^2 = \sinh^2 r = 50$; (*c*) $s^2 = 64$ while $\sinh^2 r = 36$.

### 2.5. *Phase-shift measurement with displaced squeezed states*

To understand more clearly how sensitive the displaced squeezed states (15) are with respect to phase shifts, we shall evaluate the overlap between two identical displaced squeezed states $|\Phi(\theta)\rangle$ and $|\Phi(\phi)\rangle$ which are mutually shifted by a phase $\theta-\phi$:

$$|\langle \Phi(\theta)|\Phi(\phi)\rangle|^2 = \frac{1}{[1+(\sigma_1^2-\sigma_2^2)^2 \sin^2(\theta-\phi)]^{1/2}} \times \exp\left[\frac{s^2}{4[(c_\theta^2+c_\phi^2)\sigma_2^2+(s_\theta^2+s_\phi^2)\sigma_1^2]}\left(-8(s_\theta^2+s_\phi^2)-\frac{(c_\theta-c_\phi)^2}{\sigma_1^4} + \frac{[(s_\theta+s_\phi)[1+\cos(\theta-\phi)]\sigma_1^2-(c_\theta-c_\phi)\sin(\theta-\phi)\,\sigma_2^2]^2}{\sigma_1^4[1+(\sigma_1^2-\sigma_2^2)^2\sin^2(\theta-\phi)]}\right)\right], \qquad (31)$$

where

$$\sigma_1^2 = \tfrac{1}{2}\exp(2r), \qquad \sigma_2^2 = \tfrac{1}{2}\exp(-2r), \tag{32 a}$$

and

$$c_\theta = \cos\theta, \qquad s_\theta = \sin\theta, \qquad c_\phi = \cos\phi, \qquad s_\phi = \sin\phi. \tag{32 b}$$

In the limit of $r \to 0$ we obtain from equation (31) the result for a phase distribution (see equation (22)) of a coherent state $|\xi \exp(i\phi)\rangle$ when the phase state is approximated by a coherent state $|s \exp(i\theta)\rangle$ (in addition we assume that $s = \xi$):

$$\lim_{r\to 0} |\langle \Phi(\theta)|\Phi(\phi)\rangle|^2 = \exp\{-2s^2[1-\cos(\theta-\phi)]\} = |\langle s \exp(i\theta)|s \exp(i\phi)\rangle|^2. \tag{33}$$

This single-peaked function is plotted in figure 3 ($a$). As shown in [1] in this case the phase shift can be measured with an accuracy proportional to $1/s$. In the limit of $s = 0$ we find from equation (31) the expression

$$\lim_{s\to 0} |\langle \Phi(\theta)|\Phi(\phi)\rangle|^2 = \frac{1}{[1+(\sigma_1^2-\sigma_2^2)^2 \sin^2(\theta-\phi)]^{1/2}} = |\langle r \exp(2i\theta)|r \exp(2i\phi)\rangle|^2, \tag{34}$$

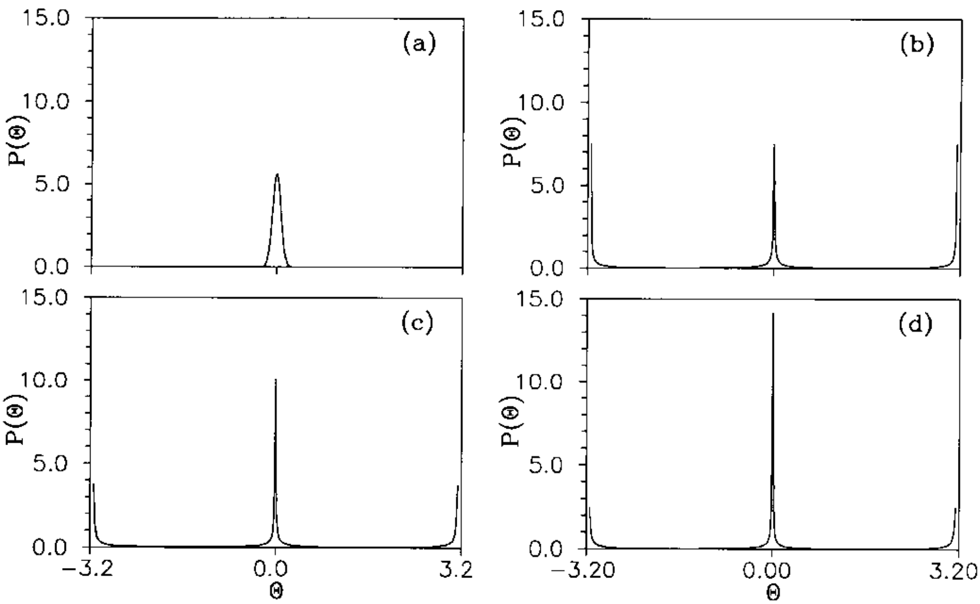


Figure 3. The phase distribution $P(\theta; s, r)$ ($\phi$ is assumed to be fixed and equal to zero), where it is assumed that the total number of photons of two overlapping 'phase' states $|\Phi(\theta)\rangle$ and $|\Phi(\phi)\rangle$ is fixed and equal to 100 (i.e. $s^2 + \sinh^2 r = 100$): ($a$) $s^2 = 100$ while $\sinh^2 r = 0$ and so from the phase distribution under consideration the phase shift measured with the help of two mutually shifted coherent states can be estimated; ($b$) $s^2 = 0$ while $\sinh^2 r = 100$ (in this case an overlap between two squeezed vacua is considered; this is a prototype of the Vogel–Schleich phase distribution); ($c$) $s^2 = \sinh^2 r = 50$; ($d$) $s^2 = 64$ and $\sinh^2 r = 36$.

which describes an overlap between two squeezed vacua which are mutually shifted by an angle $\theta-\phi$. As seen from figure 3 (*b*) the phase distribution corresponding to equation (34) has two peaks around $\theta=\phi$ and $\theta=\phi+\pi$. In figure 3 (*c*) we plot the phase distribution when the displaced squeezed states are considered with $\sinh^2 r=s^2=50$. From this figure we see that the corresponding distribution $P(\theta;\, s,\, r)$ is around $\theta=\phi$ as narrow as in the case of squeezed vacua (compare with figure 3 (*b*)). On the other hand, contributions from tails around the phase $\theta=\phi+\pi$ are suppressed. This suppression is even better seen when in figure 3 (*d*) the number of squeezed photons is smaller than the number of displaced photons. In particular, in figure 3 (*d*) we consider $\sinh^2 r=36$ and $s^2=64$.

## 3. Measurement of phase distribution based on displaced squeezed states

We now want to show how the phase distribution based on displaced squeezed states can be measured. The quantity that we are interested in finding, for a given state $|\Psi\rangle$, is

$$|\langle\Psi|\hat{U}(\phi)\hat{D}(s)\hat{S}(r)|0\rangle|^2=|\langle 0|\hat{S}(-r)\hat{D}(-s)\hat{U}(-\phi)|\Psi\rangle|^2. \tag{35}$$

The right-hand side of this equation suggests how the measurement can be accomplished. The state $|\Psi\rangle$ enters a phase shifter which shifts its phase by $-\phi$. It is displaced and then sent through a degenerate parametric amplifier where it is squeezed. Finally, the probability that the transformed state has zero photons in it is determined by photocount measurements. This is depicted in figure 4.

The shift can be accomplished by means of a beam splitter and a large-amplitude coherent state [17]. Let us suppose that the beam splitter is described by a transformation

$$\begin{bmatrix}\hat{a}_{1,\,\text{out}}\\ \hat{a}_{2,\,\text{out}}\end{bmatrix}=\begin{bmatrix}T^{1/2} & -R^{1/2}\\ R^{1/2} & T^{1/2}\end{bmatrix}=\begin{bmatrix}\hat{a}_{1,\,\text{in}}\\ \hat{a}_{2,\,\text{in}}\end{bmatrix}. \tag{36}$$

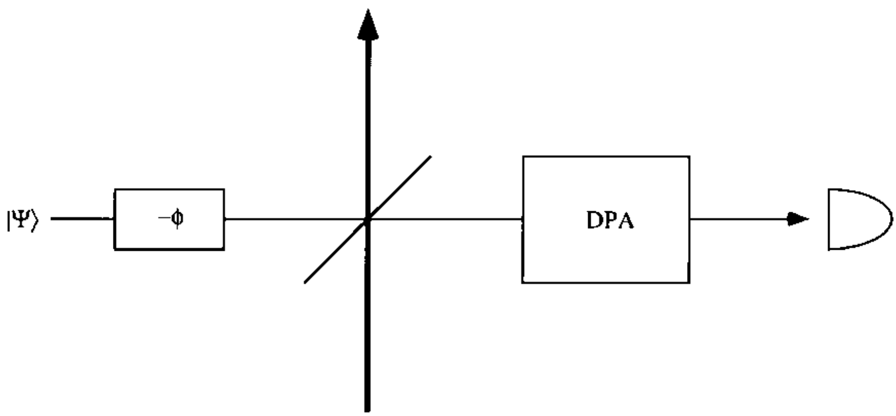


Figure 4. Method for measuring the phase distribution. The state whose distribution is to be measured is first sent through a phase shifter, next to a beam splitter where it is mixed with a strong local oscillator, and then through a degenerate parametric amplifier (DPA). The resulting signal is measured with a photodetector, and the probability of detecting no photons is determined. This probability gives the overlap of the original state with a displaced squeezed state. The phase distribution is constructed by repeating this procedure for different values of $\phi$.

If the input state is $|\Psi\rangle_1|s'\rangle_2$, that is mode 1 is in the state $|\Psi\rangle_1$ and mode 2 is in the coherent state $|s'\rangle_2 = \hat{D}_2(s')|0\rangle_2$ with amplitude $s'$, then, if $R \ll 1$ and $|s' R^{1/2}| \ll 1$, the output state in the mode 1 is approximately $\hat{D}_1(-R^{1/2}s)|\Psi\rangle_1$. If $s' = s/R^{1/2}$, we obtain the desired displacement. It is this procedure which allows us to call our phase distribution operational.

## 4. Properties of phase distribution

In this section we would like to discuss some properties of our phase distribution which make it useful in determining the utility of a state for phase-shift measurements. A phase-shift measurement is made by sending a state $|\Psi\rangle$ through a phase shifter. The output state from this phase shifter is $\hat{U}(\phi)|\Psi\rangle$. We want to determine $\phi$ by comparing $|\Psi\rangle$ and $\hat{U}(\phi)|\Psi\rangle$. If the states are very different, it should be possible to determine $\phi$ with considerable precision. As a measure of the distinguishability of these two states we use the inner product [18, 19]

$$F(\phi) = |\langle\Psi|\hat{U}(\phi)|\Psi\rangle|. \tag{37}$$

This function has a maximum at $\phi = 0$. The width of this maximum gives us an estimate of the accuracy with which $\phi$ can be determined. Thus we want to determine what $P(\phi)$ can tell us about $F(\phi)$. Let us first examine the behaviour of $F(\phi)$ near $\phi = 0$. The first derivative of $F(\phi)$ vanishes at $\phi = 0$ while the second derivative is $-(\Delta N)^2$. Therefore near $\phi = 0$ we can approximate $F(\phi)$ as

$$F(\phi) \approx 1 - \tfrac{1}{2}(\Delta N)^2\phi^2. \tag{38}$$

The phase distribution defined by equation (19) is of the form (in what follows we omit the parameters $s$ and $r$ from the notation for the phase distribution)

$$P(\theta) = \mathcal{N}\,|\langle\Psi|\hat{U}(\theta)|\Phi\rangle|^2, \tag{39}$$

where $\mathcal{N}$ is a normalization constant and $|\Phi\rangle$ is the state from which the phase distribution is generated. In section 2 it was taken to be a displaced squeezed state. We can compare $|\Psi\rangle$ with $\hat{U}(\phi)|\Psi\rangle$ by comparing $P(\theta)$ with $P(\theta-\phi)$. In particular, if we define

$$G(\phi) = \int_0^{2\pi} d\theta\, |\langle\Psi|\hat{U}(\theta)|\Phi\rangle|\,|\langle\Psi|\hat{U}^\dagger(\phi)\hat{U}(\theta)|\Phi\rangle|, \tag{40}$$

then

$$\left.\frac{dG(\phi)}{d\phi}\right|_{\phi=0} = 0, \tag{41}$$

$$\begin{aligned}\left.\frac{d^2G(\phi)}{d\phi^2}\right|_{\phi=0} &= -\int_0^{2\pi} d\theta \left(\frac{d}{d\theta}|\langle\Psi|\hat{U}^\dagger(\phi)\hat{U}(\theta)|\Phi\rangle|_{\phi=0}\right)^2 \\ &= \frac{1}{4}\int_0^{2\pi} d\theta\, \frac{(\langle\Psi|[\hat{N}, \hat{Q}(\theta)]|\Psi\rangle)^2}{\langle\Psi|\hat{Q}(\theta)|\Psi\rangle},\end{aligned} \tag{42}$$

where

$$\hat{Q}(\theta) = \hat{U}(\theta)|\Phi\rangle\langle\Phi|\hat{U}^\dagger(\theta) = |\Phi(\theta)\rangle\langle\Phi(\theta)|. \tag{43}$$

From these equations it is clear that $G(\phi)$ has a maximum at $\phi = 0$. It would be useful to find a relation between the second derivative of $G(\phi)$ at $\phi = 0$ to that of $F(\phi)$ at zero. This would allow us to compare the local behaviour of $G(\phi)$ and $F(\phi)$ near their maximum at zero. We begin by noting that, if we define $\Delta\hat{N} \equiv \hat{N} - \langle\hat{N}\rangle$, then

$$[\Delta\hat{N}, \hat{Q}(\theta)] = [\hat{N}, \hat{Q}(\theta)]. \tag{44}$$

We then can apply the Schwarz inequality to give

$$|\langle\Psi|[\Delta\hat{N}, \hat{Q}(\theta)]|\Psi\rangle| \leqslant 2(\Delta N)[\langle\Psi|\hat{Q}(\theta)|\Psi\rangle]^{1/2}. \tag{45}$$

Finally, we can use the fact that $\langle\Psi|[\Delta\hat{N}, \hat{Q}(\theta)]|\Psi\rangle$ is imaginary to give

$$(\langle\Psi|[\Delta\hat{N}, \hat{Q}(\theta)]|\Psi\rangle)^2 = -|\langle\Psi|[\Delta\hat{N}, \hat{Q}(\theta)]|\Psi\rangle|^2 \geqslant -4(\Delta N)^2\langle\Psi|\hat{Q}(\theta)|\Psi\rangle, \tag{46}$$

which give us

$$\frac{1}{2\pi}\frac{\mathrm{d}^2G(\phi)}{\mathrm{d}\phi^2}\bigg|_{\phi=0} \geqslant -(\Delta N)^2. \tag{47}$$

This implies that the maximum of the function $(1/2\pi)G(\phi)$ is broader than that of $F(\phi)$. Therefore, by measuring $G(\phi)$, we can obtain a limit on how fast $F(\phi)$ falls off near $\phi = 0$; that is it must fall off at least as fast as $(1/2\pi)G(\phi)$.

We can supplement the local information about the peak with some global information. This is useful because, although the peak may fall off rapidly near $\phi = 0$, it may have large tails which limit the accuracy to which $\phi$ can be measured. We can obtain this information by using the inequality (which is proved in appendix A):

$$|\langle\Psi_1|\Xi\rangle\langle\Xi|\Psi_2\rangle| \leqslant \tfrac{1}{2}(1 + |\langle\Psi_1|\Psi_2\rangle|), \tag{48}$$

where $|\Psi_1\rangle$, $|\Psi_2\rangle$ and $|\Xi\rangle$ are any three vectors of norm 1 in Hilbert space. Choosing $|\Psi_1\rangle = |\Psi\rangle$, $|\Psi_2\rangle = \hat{U}(\phi)|\Psi\rangle$ and $|\Xi\rangle = \hat{U}(\theta)|\Phi\rangle$, and taking the maximum over all $\theta$ we have

$$\sup_{\theta}|\langle\Psi|\hat{U}(\theta)|\Phi\rangle\langle\Phi|\hat{U}^{\dagger}(\theta)\hat{U}(\phi)|\Psi\rangle| \leqslant \tfrac{1}{2}[1 + F(\phi)]. \tag{49}$$

This relation provides us with a lower bound on $F(\phi)$ in terms of our operational phase distribution. Note that this bound is useful only if the left-hand side of equation (49) is greater than $\frac{1}{2}$.

The inequality (49) can give us information about the peak in $F(\phi)$ at $\phi = 0$ if the overlap between $|\Psi\rangle$ and $\hat{U}(\theta')|\Phi\rangle$ is large for some values of $\theta'$, that is $|\langle\Psi|\hat{U}(\theta')|\Phi\rangle|$ is close to unity. For $\phi$ not too large the left-hand side of equation (49) will then provide us with a useful lower bound for the peak of $F(\phi)$.

As a simple example let us consider the case when $|\Psi\rangle$ is the coherent state $|s\rangle$, where $s = |s| \exp(\mathrm{i}\eta)$, and $|\Phi\rangle$ is the coherent state with a real amplitude $|s|$. We find that, for $|s| \ll 1$ (see equation (33) where the exact expression for $|F(\phi)|^2$ with $\theta = 0$ is presented):

$$F(\phi) \approx \exp\left(-\frac{|s|^2\phi^2}{2}\right), \tag{50}$$

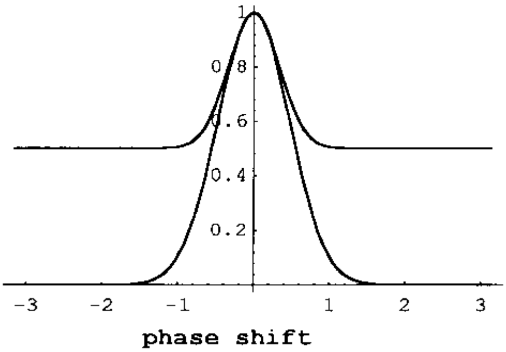


Figure 5. The two sides of the inequality in equation (49) plotted for a coherent state with a mean photon number of 9 ($s = 3$). The upper curve is the right-hand side, and the lower curve is the left-hand side. We see that the information provided by the measureable phase distribution (lower curve) gives a good description of the peak of $F(\phi)$ near $\phi = 0$.

and

$$\sup_{\theta} |\langle \Psi| \hat{U}(\theta) |\Phi\rangle \langle \Phi| \hat{U}^{\dagger}(\theta) \hat{U}(\phi) |\Psi\rangle| \approx \exp\left(-\frac{|s|^2 \phi^2}{4}\right). \tag{51}$$

These quantities are then inserted into the left- and right-hand sides of equation (48) and the results are plotted in figure 5. We see that we can obtain useful information about $F(\phi)$ from measured quantity in equation (51).

Finally, let us determine when our phase distribution of a given state is approximately the same as the London phase distribution [4, 5] of that state. The London phase distribution is based on the unnormalized phase states

$$|\theta\rangle = \frac{1}{(2\pi)^{1/2}} \sum_{n=0}^{\infty} \exp(\mathrm{i}n\theta) \, |n\rangle, \tag{52}$$

and for the state $|\Psi\rangle$ is given by

$$P_{\mathrm{L}}(\theta) = |\langle \theta | \Psi \rangle|^2. \tag{53}$$

This distribution is normalized as it stands. We shall compare $P(\theta)$ and $P_{\mathrm{L}}(\theta)$ by examining the number state expansions of $|\theta\rangle$ and $\hat{U}(\theta)|\Phi\rangle$, where $|\Phi\rangle = \hat{D}(s)\hat{S}(r)|0\rangle$. We begin by considering the matrix elements $\langle n|\Phi\rangle$. We have that [15]

$$\langle n|\Phi\rangle = \left(\frac{\operatorname{sech} r}{n!}\right)^{1/2} \exp\left[-\tfrac{1}{2}s^2 \exp(-2r)\,(1 + \tanh r)\right](-\mathrm{i})^n \times \left(\frac{\tanh r}{2}\right)^{n/2} H_n\left(\mathrm{i}\,\frac{s \exp(-r) \operatorname{sech} r}{(2 \tanh r)^{1/2}}\right), \tag{54}$$

where $H_n$ is the $n$th Hermite polynomial. In order to proceed, we need to consider in what regime we are working. Let us assume that the states $|\Psi\rangle$ for which we are computing the phase distribution satisfy the condition

$$|\langle \Psi| \hat{a} |\Psi\rangle| \ll (\Delta N)^{1/2}. \tag{55}$$

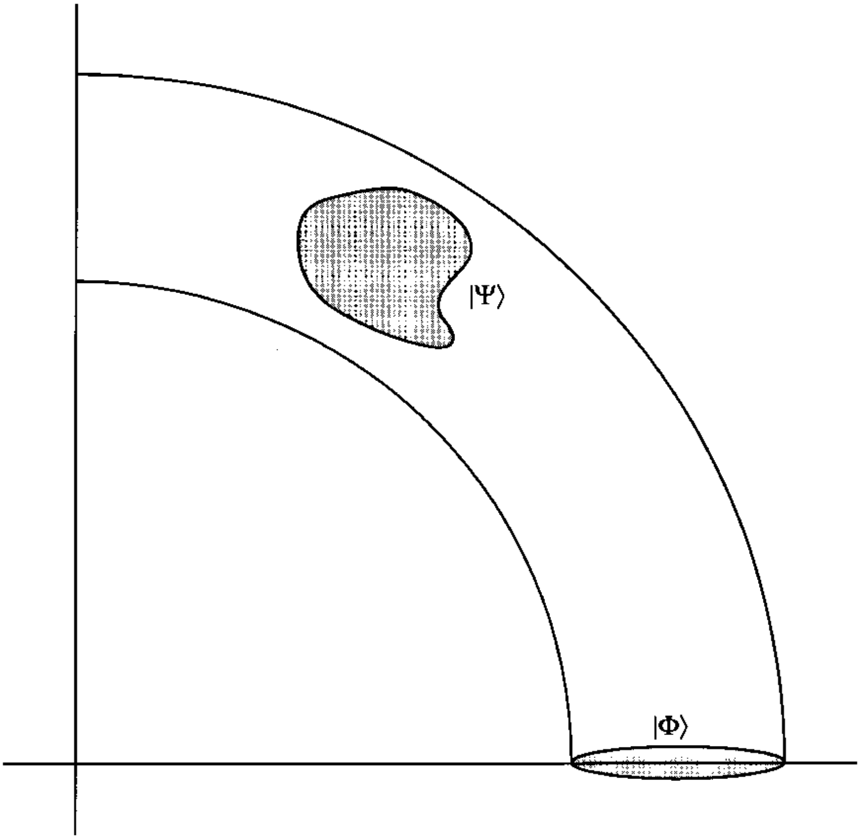


Figure 6. Phase-space representations of the state that we wish to measure, $|\Psi\rangle$, and the state on which the phase distribution is based, $|\Phi\rangle$.

This means that in phase space $|\Psi\rangle$ is represented by a blob which is contained in a circular band whose centre is the origin of the phase space and whose width is much smaller than its radius (figure 6). We want our state $|\Phi\rangle$, on which our phase distribution will be based, to fit roughly inside this circular band. The state $|\Phi\rangle$ is centred at the point $s$ of phase space and its width is $(\exp r)/2$. We are therefore interested in the regime

$$s \ll \exp r. \tag{56}$$

If this condition is satisfied, then the number states $|n\rangle$ which will have a substantial overlap with $|\Phi\rangle$ will be those for which $n^{1/2} \approx s$. On the other hand, the magnitude of the argument of the Hermite polynomial, which we shall denote by $\xi$, is approximately given by

$$\xi \approx 2^{1/2} \exp(-2r), \tag{57}$$

for $r$ of order one or greater. This means that for the Hermite polynomials we shall be interested in the regime $n \ll \xi$ and we can use an appropriate asymptotic expansion for $H_n$. The details of this calculation are given in appendix B.

The result is

$$\langle n|\Phi\rangle \approx \frac{(\operatorname{sech} r)^{1/2}}{2} \exp\left[-\tfrac{1}{2}s^2 \exp(-2r)(1+\tanh r)\right] \times \exp\left[-\xi^2 + (2n)^{1/2}\xi\right]\left(\frac{2}{\pi n}\right)^{1/4} (\tanh r)^{n/2}. \tag{58}$$

We now want to look at this expression for $n$ in the neighbourhood of $s^2$. In order to do so we first express the $n$-dependent part as an exponential

$$\left(\frac{1}{n}\right)^{1/4} \exp\left[(2n)^{1/2}\xi\right](\tanh r)^{n/2} = \exp\left[g(n)\right], \tag{59}$$

where

$$g(n) \approx \tfrac{1}{4}\ln n + \frac{n}{2}\ln\left[1 - 2\exp(-2r)\right] + (2n)^{1/2}\xi. \tag{60}$$

The function $g(n)$ has a maximum at

$$n \approx 2\left(\frac{\xi}{|\ln[1-2\exp(-2r)]|}\right)^2 \approx s^2. \tag{61}$$

Expanding $g(n)$ about this maximum, we find that

$$g(s^2 + \delta n) \approx g(s^2) - \frac{1}{4s^2}(\delta n)^2 \exp(-2r). \tag{62}$$

This implies that the width of the peak at $s^2$ is approximately given by $s\exp r$. Therefore, if $\delta n \ll s\exp r$, then $g(s^2+\delta n) = g(s^2)$. This implies that, for $|n - s^2| \ll s\exp r$, the matrix element $\langle n|\Phi\rangle$ is approximately independent of $n$. For $n$ in this range the state $|\Phi\rangle$ 'looks like' the phase state $|\theta = 0\rangle$ which is given by equation (52).

This means that, for some states $|\Psi\rangle$, the phase distributions $P(\theta)$ and $P_L(\theta)$ are almost the same. Let us suppose that $|\langle n|\Phi\rangle|$ is appreciable only for $n_1 < n < n_2$ where $n_1 \ll 1$ and $n_2 - n_1 \ll n_1$. If we choose a displaced squeezed state $|\Phi\rangle$ given by equation (15 $b$) such that $s = ((n_1)^{1/2} + (n_2)^{1/2})/2$, $n_2 - n_1 \ll s\exp r$, and $|\langle 0|\Phi\rangle| \ll 1$, then the two phase distributions will be approximately the same. This provides a method of measuring the London distribution for states which satisfy these conditions.

## 5. Conclusions

At present there are two operational phase distributions. The first was proposed by Noh *et al*. [11] and is based on an eight-port homodyne measurement. This approach has been shown to be equivalent to measuring the integrated $Q$ function of a state [20, 21]. The second is the scheme of Vogel and Schleich [10] which is based on quadrature eigenstates.

Each of these methods has its limitations. The Noh *et al.* scheme measures a rather noisy phase distribution. This is a result of open ports in their measuring device which mix the vacuum noise with the state whose distribution is being determined. The result is a smeared phase distribution which can obscure the quantum-mechanical properties of a state. The Vogel–Schleich scheme is limited to states whose angular width in phase space is not too great. For example, we saw that the Vogel–Schleich distribution for an odd coherent state does not exist.

The operational phase distribution based on displaced squeezed states overcomes some of these difficulties. It will not be as noisy as the distribution resulting from the method of Noh *et al.* and it will be capable of providing phase distributions for states for which the Vogel–Schleich distribution does not work.

Unlike the other two methods the measurement scheme presented in this paper is state dependent. Our phase states are displaced squeezed states. One wants to pick the squeezing and displacement parameters so that the state which is being measured is 'covered' by the squeezing ellipses which are acting as the phase states. As we saw when considering the measurement of a coherent state, it often does not take much squeezing to sharpen a phase distribution significantly. This is consistent with the work of Freyberger and Schleich [22]. They examined the phase width of a displaced squeezed state and found that for a fixed number of photons in the state the width is smallest when there is considerably more displacement than squeezing.

The phase distribution presented in this paper is also of use in determining whether a state is useful for detecting small phase shifts. This suggests that it could be useful in the analysis of interferometric measurements.

We were also able to relate our operational phase distribution to the London or Pegg–Barnett phase distribution. We found that under certain conditions they are identical. Because the phase distribution based on displaced squeezed states is measurable it can, for some states, provide a means of actually measuring their London phase distribution.

In summary, the operational phase distribution presented here has a number of useful properties; we believe that it can be useful in the measurement of phase shifts and phase differences. We are at present studying these possibilities.

**Acknowledgments**

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## Appendix A

We wish to prove the inequality in equation (48), that is, if $|\Psi_1\rangle$, $|\Psi_2\rangle$ and $|\Xi\rangle$ are three vectors in a Hilbert space of unit norm, then

$$|\langle\Psi_1|\Xi\rangle\langle\Xi|\Psi_2\rangle| \leqslant \tfrac{1}{2}(1+|\langle\Psi_1|\Psi_2\rangle|). \tag{A 1}$$

We first note that, if $|\Xi\rangle$ is expressed as

$$|\Xi\rangle = |\Xi_\parallel\rangle + |\Xi_\perp\rangle, \tag{A 2}$$

where $|\Xi_\parallel\rangle$ is a linear combination of the vectors $|\Psi_1\rangle$ and $|\Psi_2\rangle$, while $|\Xi_\perp\rangle$ is orthogonal to both of these vectors, that

$$|\langle\Psi_1|\Xi\rangle\langle\Xi|\Psi_2\rangle| = |\langle\Psi_1|\Xi_\parallel\rangle\langle\Xi_\parallel|\Psi_2\rangle| \leqslant |\langle\Psi_1|\Xi_\parallel\rangle\langle\Xi_\parallel|\Psi_2\rangle|, \tag{A 3}$$

where $|\Xi_\parallel\rangle = |\Xi_\parallel\rangle/\langle\Xi_\parallel|\Xi_\parallel\rangle$. This allows us to conclude that, if equation (A 1) is valid for all vectors of norm one which are linear combinations of $|\Psi_1\rangle$ and $|\Psi_2\rangle$, then it is true for all vectors of norm one.

We shall now assume that $|\Xi\rangle$ is a norm-one vector in the space spanned by $|\Psi_1\rangle$ and $|\Psi_2\rangle$. Define the orthogonal unit vectors

$$|\upsilon_1\rangle = \frac{\langle\Psi_1|\Psi_2\rangle}{|\langle\Psi_1|\Psi_2\rangle|}|\Psi_1\rangle, \qquad |\upsilon_2\rangle = \frac{1}{(1-|\langle\Psi_2|\Psi_2\rangle|^2)^{1/2}}(|\Psi_2\rangle - |\Psi_1\rangle\langle\Psi_1|\Psi_2\rangle). \tag{A 4}$$

Then the state $|\Psi_2\rangle$ can be expressed as

$$|\Psi_2\rangle = |\langle\Psi_1|\Psi_2\rangle||\upsilon_1\rangle + (1-|\langle\Psi_1|\Psi_2\rangle|^2)^{1/2}|\upsilon_2\rangle, \tag{A 5}$$

and we can express $|\Xi\rangle$ as

$$|\Xi\rangle = c_1|\upsilon_1\rangle + c_2|\upsilon_2\rangle, \tag{A 6}$$

where $|c_1|^2 + |c_2|^2 = 1$. The left-hand side of equation (A 1) becomes

$$|\langle\Psi_1|\Xi\rangle\langle\Xi|\Psi_2\rangle| = |c_1|\left|c_1|\langle\Psi_1|\Psi_2\rangle| + c_2(1-|\langle\Psi_1|\Psi_2\rangle|^2)^{1/2}\right|. \tag{A 7}$$

From this equation it is clear that this expression will attain its largest value when $c_1$ and $c_2$ have the same phase. We shall choose them both to be real and positive. Let us set

$$c_1 = \cos\gamma, \qquad c_2 = \sin\gamma, \tag{A 8}$$

where $0 \leqslant \gamma \leqslant \pi/2$, and, in addition, let us define an angle $\beta$ by

$$\cos\beta = |\langle\Psi_1|\Psi_2\rangle|. \tag{A 9}$$

With these definitions we have

$$|\langle\Psi_1|\Xi\rangle\langle\Xi|\Psi_2\rangle| = \cos\gamma\cos(\gamma-\beta). \tag{A 10}$$

We can now find the maximum of this expression considered as a function of $\gamma$. The maximum value occurs at $\gamma = \beta/2$ and is equal to the right-hand side of equation (A 1). This proves our result.

## Appendix B

We want to find an expression for $H_n(\mathrm{i}x)$, where $x$ is real and $n \ll x$. We start from the integral representation [23]

$$\begin{aligned} H_n(\mathrm{i}x) &= \frac{(2\mathrm{i})^n}{\pi^{1/2}}\int_{-\infty}^{\infty} \mathrm{d}t\,(x+t)^n \exp(-t^2) \\ &= \frac{(2\mathrm{i})^n}{\pi^{1/2}} n^{(n+1)/2}\int_{-\infty}^{\infty}\mathrm{d}z\left(\frac{x}{n^{1/2}+z}\right)^n \exp(-nz^2), \end{aligned} \tag{B 1}$$

where $z = t/n^{1/2}$. In order to find an asymptotic form valid for large $n$, Laplace's method can be applied to the integral in equation (B 1) [24]. This is most easily accomplished if the integral is expressed in the form

$$\begin{aligned} \int_{-\infty}^{\infty}\mathrm{d}z\left(\frac{x}{n^{1/2}+z}\right)^n \exp(-nz^2) = (-1)^n &\int_{-\infty}^{-x/n^{1/2}} \mathrm{d}z \exp[n\Phi_-(z)] \\ &+ \int_{-x/n^{1/2}}^{\infty} \mathrm{d}z\exp[n\Phi_+(z)], \end{aligned} \tag{B 2}$$

where

$$\Phi_-(z) = \ln\left(-\frac{x}{n^{1/2}} - z\right) - z^2, \qquad \Phi_+(z) = \ln\left(\frac{x}{n^{1/2}} + z\right) - z^2. \tag{B 3}$$

The dominant contributions for large $n$ are given by the maximum of $\Phi_+(z)$ at $z_+$, where

$$z_+ = \frac{1}{2}\left[-\frac{x}{n^{1/2}} + \left(\frac{x^2}{n} + 2\right)^{1/2}\right], \tag{B 4}$$

and the maximum of $\Phi_-(z)$ at $z_-$, where

$$z_- = \frac{1}{2}\left[-\frac{x}{n^{1/2}} - \left(\frac{x^2}{n} + 2\right)^{1/2}\right]. \tag{B 5}$$

These two contributions gives us the result

$$H_n(\mathrm{i}x) \approx \frac{(2\mathrm{i})^n}{2^{1/2}}\left(\frac{n}{2}\right)^{n/2} \exp\left(-\frac{(n+x^2)}{2}\right)\{\exp[x(2n)^{1/2}] + (-1)^n \exp[-x(2n)^{1/2}]\}. \tag{B 6}$$

Substitution of this result into equation (54) and application of the Stirling approximation to the $1/(n!)^{1/2}$ factor yields equation (58).

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