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Reconstruction of Wigner Functions on Different Observation Levels

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We present a method for a reconstruction of Wigner functions of quantum mechanical states of light on different observation levels. Using the Jaynes principle of Maximum Entropy we show how to reconstruct the Wigner function on the given observation level which is characterized by mean values of a set of observables. We present examples illustrating the power of the proposed method. In particular, we analyze the reconstruction of Wigner functions of coherent states, squeezed states, Fock states, and superpositions of coherent states on various observation levels of physical relevance. © 1996 Academic Press, Inc.

1. INTRODUCTION

The concept of a *state* of a physical system constitutes one of the most important building stones of any physical theory. In classical physics the state of a system can be associated with a "point" in a corresponding phase space. Dynamical evolution of a classical system is then described as a trajectory in this phase space. In classical physics a point in the phase space can be "located" with arbitrary accuracy, and, in principle, the state of an individual system can be directly measured [1]. Alternatively, in the classical statistical mechanics the state of a system can be described in terms of a probability density distribution in the phase space in which case only a probability that the system is in a particular region of the phase space can be determined. Nevertheless, there are no physical reasons in classical physics, why the state cannot be identified with a point in the phase space.

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Definition of a state in quantum physics is more abstract and complex [2]. Operationally, a state of a quantum-mechanical system is associated with particular probability distributions of measured physical observables. These distributions are obtained via measurements over an ensemble of quantum-mechanical systems which are prepared in the same way (i.e., they are in the same state). Formally, the state of the quantum-mechanical system is described either as a vector in a Hilbert space (in the case of pure states) or by a density operator. Equivalently, the state of a quantum-mechanical system can be described by a wave-function or, in the framework of the phase-space formalism [3], the state under consideration can be described with the help of phase-space quasiprobability density distributions.

As we have said earlier, classical dynamical variables can be measured to arbitrary accuracy in principle. This permits precise measurement of conjugated variables such as position and momentum, and allows joint probability density distribution to be constructed for a phase-space description of dynamics. The lack of commutability of conjugated observables in quantum mechanics leads to the fact that the "point" in the quantum-mechanical phase space cannot be localized precisely, i.e., there is always a fundamental limit with which this "point" can be determined. Another consequence of non-commutability of conjugated observables is the lack of a unique rule by which quantum and classical variables are associated. This results into a number of (quasi)probability density distributions associated with a phase-space description of a quantum-mechanical state. Depending on the operator ordering a number of different (quasi)probability density distributions can be defined of which the best known are the Wigner function [4], the Husimi (Q) function [5], and the Glauber-Sudarshan (P) function [6], reflecting the symmetric (Weyl), antinormal and normal ordering of operators in the corresponding characteristic function [7]. The P function can be singular or negative, the Wigner function can be negative but is regular, whereas the Q function is always non-negative and regular [5, 8, 9]. We note that all (quasi)probability density distributions under consideration contain *complete* information about the state of the system. Cahil and Glauber [7] have shown that all these can be contained in an s-parameterized quasiprobability density distribution where the choice of the parameter s determines the degree of "smoothing" from the P function (in this case $s=1$) to the Q function ($s=-1$), while for the Wigner function $s=0$.

The Wigner function plays an exceptional role among all quasiprobability density distributions. Firstly, it generates proper marginal distributions for individual phase-space variables. Secondly, under the action of linear canonical transformations the Wigner function behaves exactly in the same way as the classical probability density distributions [10]. The Wigner function contains *complete* information about the state of the system, i.e., it carries the same information as the density operator or the corresponding wave function. From the Wigner function one can evaluate all (symmetrically-ordered) moments of the system operators. On the other hand, the inverse is also valid. It means that from the knowledge of the *complete* set of moments of system operators the Wigner function (as well as the density operator) can be determined uniquely [11]. Because of these properties (see also

[4]) we will concentrate our attention on the problem of a reconstruction of the Wigner function of a quantum-mechanical state. In particular, we will consider the Wigner function of a single-mode quantum electromagnetic field described as a harmonic oscillator.

It is well known that the wave-function of a quantum-mechanical system cannot be measured directly [12]. A *single* measurement does not yield enough information which allows us to determine the state of the system uniquely [12]. In addition, due to the fact that conjugated observables do not commute, the quantum-mechanical measurement inevitably disturbs the state, so the information about the conjugated observable cannot be obtained from subsequent measurements. Analogously, one cannot measure directly the Wigner function of the quantum-mechanical system. On the other hand, the complete information about the state can be obtained if one performs a *sufficient* number of measurements on different members of an ensemble of *identically* prepared states of the quantum system under consideration [12]. From here it follows that the Wigner function of a quantum-mechanical state can, in principle, be reconstructed.

We can consider two different schemes for reconstruction of the Wigner function of the quantum-mechanical state $|\Psi\rangle$. The difference between these two schemes is based on the way in which the information about the quantum-mechanical system is obtained. One can either perform a measurement of each observable *independently* or one can consider a *simultaneous* measurement of conjugated observables (in both cases we assume an ideal, i.e., the unit-efficiency, measurement).

In the first kind of the measurement a *distribution* $W_{|\Psi\rangle}(A)$ for a particular observable $\hat{A}$ in the state $|\Psi\rangle$ is measured in an unbiased way [13], i.e., $W_{|\Psi\rangle}(A) = |\langle \Phi_A \mid \Psi \rangle|^2$, where $|\Phi_A\rangle$ are eigenstates of the observable $\hat{A}$ such that $\sum_A |\Phi_A\rangle\langle\Phi_A| = \hat{1}$. Here a question arises: What is the *smallest* number of distributions $W_{|\Psi\rangle}(A)$ required to determine the state uniquely? This question is directly related to the so-called Pauli problem [14] of the reconstruction of the wave-function from distributions $W_{|\Psi\rangle}(q)$ and $W_{|\Psi\rangle}(p)$ for the position and momentum of the state $|\Psi\rangle$. As shown by Gale, Guth and Trammel [15], in general, the knowledge of $W_{|\Psi\rangle}(q)$ and $W_{|\Psi\rangle}(p)$ is not sufficient for a complete reconstruction of the wave (Wigner) function. In contrast, one can consider an *infinite* set of distributions $W_{|\Psi\rangle}(x_\theta)$ of the rotated quadratures $\hat{x}_\theta = \hat{q}\cos\theta + \hat{p}\sin\theta$. Each distribution $W_{|\Psi\rangle}(x_\theta)$ can be obtained in a measurement of a *single* observable $\hat{x}_\theta$ in which case a detector (filter) is prepared in an eigenstate $|x_\theta\rangle$ of this observable. It has been shown by Vogel and Risken [16] that from an infinite set of the measured distributions $W_{|\Psi\rangle}(x_\theta)$ for all values of θ such that $[0 < \theta \leqslant \pi]$ the Wigner function can be reconstructed uniquely via the inverse Radon transformation. In other words the knowledge of the set of distributions $W_{|\Psi\rangle}(x_\theta)$ is equivalent to the knowledge of the Wigner function. This scheme of reconstruction of the Wigner function (the so called optical homodyne tomography) has recently been experimentally realized by Raymer and his coworkers [17] and the Wigner function of a coherent state and a squeezed vacuum state have been experimentally reconstructed.

In the case of the simultaneous measurement of two non-commuting observables (let say $\hat{q}$ and $\hat{p}$) it is not possible to construct an eigenstate of these two operators, and therefore it is inevitable that the simultaneous measurement of two non-commuting observables introduces additional noise (of quantum origin) into measured data [5, 8, 9, 18, 19]. To describe a process of a simultaneous measurement of two non-commuting observables Wódkiewicz [18] has proposed a formalism based on an operational probability density distribution which explicitly takes into account the action of the measurement device modelled a "filter" (quantum ruler). A particular choice of the state of the ruler samples a specific type of accessible information concerning the system, i.e., information about the system is biased by the filtering process. The quantum-mechanical noise induced by filtering formally results into smoothing of the original Wigner function of the measured state [5, 8], so that the operational probability density distribution can be expressed as a convolution of the original Wigner function and the Wigner function of the filter state [18]. In particular, if the filter is considered to be in a vacuum state then the corresponding operational probability density distributions is equal to the Husimi (Q) function [5]. The Q function of optical fields has been experimentally measured by Walker and Carroll [20]. The direct experimental measurement of the operational probability density distribution with the filter in an arbitrary state is feasible in an 8-port experimental setup used by Noh, Fougéres and Mandel [21]. The price to pay for the simultaneous measurement of non-commuting observables is that the measured distributions are fuzzy (i.e., they are equal to smoothed Wigner functions). Nevertheless, if detectors used in the experiment have a unit efficiency (in the case of an ideal measurement) the noise induced by quantum filtering can be "separated" from the measured data and the Wigner function can be reconstructed from the operational probability density distribution. In particular, the Wigner function can be uniquely reconstructed from the Q function.[1]

As we have already indicated it is well understood now that the Wigner function can, in principle, be reconstructed using either the single observable measurements (the optical homodyne tomography) or the simultaneous measurement of two non-commuting observables. The completely reconstructed Wigner function contains information about *all* independent moments of the system operators, i.e., in the case of the quantum harmonic oscillator the knowledge of the Wigner function is equivalent to the knowledge of all moments $\langle (\hat{a}^\dagger)^m \hat{a}^n \rangle$ of the creation ($\hat{a}^\dagger$) and annihilation ($\hat{a}$) operators.

In many cases it turns out that the state under consideration is characterized by an *infinite* number of independent moments $\langle (\hat{a}^\dagger)^m \hat{a}^n \rangle$ (for all m and n). To perform a *complete* measurement of these moments can take an infinite time. This means that even though the Wigner function can in principle be reconstructed the collection of experimental data take an infinite time. In addition the data processing

[1] We note that the "deconvolution" of the vacuum from the Q function can suffer greatly from noise in the data. Raymer *et al.* (see Ref. [17]) have proposed another more effective way to reconstruct Wigner functions from "noisy" data associated with Q functions.

and numerical reconstruction of the Wigner function are time consuming as well. Therefore experimental realization of the reconstruction of the Wigner function can be questionable.

In practice, it is possible to perform a measurement of just a finite number of independent moments of the system operators. The aim of this paper is to analyze how the Wigner function can be (partially) reconstructed from an incomplete knowledge about the system (i.e., from a finite number of moments of system operators) and how to quantify the precision with which the Wigner function is reconstructed. To accomplish this task we utilize the concept of observation levels [22] (see also [23]) where each observation level is specified by a set of linearly independent operators $\hat{G}_\nu$ ($\nu = 1, 2, ..., n$) for which expectation values G_ν are given (measured). With the help of the Jaynes principle of the maximum entropy (the so called MaxEnt principle) [24] (see also [22, 25]) we will show how to reconstruct in the most reliable way the Wigner function of the measured state within a given observation level. The paper is organized as follows: in Section 2 we briefly review basic elements of the phase-space formalism used in quantum optics. We also specify those nonclassical states which are studied later in the paper. In Section 3 we introduce concept of observation levels applied to quantum optics. In Section 4 we show how with the help of the MaxEnt principle Wigner functions on given observation levels can be reconstructed. In Section 5 we analyze Wigner functions of various nonclassical states of light on different observation levels. Section 6 is devoted to a discussion of a relation between the standard von Neumann measurement theory and the concept of observation levels. We also discuss the relation between optical homodyne tomography and measurements on various observation levels. We finish our paper with conclusions.

2. Phase-Space Description of States of Single-Mode Field

Utilizing a close analogy between the operator for the electric component $\hat{E}(r, t)$ of a monochromatic light field and the quantum-mechanical harmonic oscillator we will consider a dynamical system which is described by a pair of canonically conjugated Hermitean observables $\hat{q}$ and $\hat{p}$,

$$[\hat{q}, \hat{p}] = i\hbar. \tag{2.1}$$

Eigenvalues of these operators range continuously from $-\infty$ to $+\infty$. The annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$ can be expressed as a complex linear combination of $\hat{q}$ and $\hat{p}$

$$\hat{a} = \frac{1}{\sqrt{2h}}(\lambda\hat{q} + i\lambda^{-1}\hat{p}); \qquad \hat{a}^\dagger = \frac{1}{\sqrt{2h}}(\lambda\hat{q} - i\lambda^{-1}\hat{p}), \tag{2.2}$$

where λ is an arbitrary real parameter. The operators $\hat{a}$ and $\hat{a}^\dagger$ obey the Weyl-Heisenberg commutation relation

$$[\hat{a}, \hat{a}^\dagger] = 1, \tag{2.3}$$

and therefore possess the same algebraic properties as the operator associated with the complex amplitude of a harmonic oscillator (in this case $\lambda = \sqrt{m\omega}$, where m and ω are the mass and the frequency of the quantum-mechanical oscillator, respectively) or the photon annihilation and creation operators of a single mode of the quantum electromagnetic field. In this case $\lambda = \sqrt{\varepsilon_0 \omega}$ (ε_0 is the dielectric constant and ω is the frequency of the field mode) and the operator for the electric field reads (we do not take into account polarization of the field)

$$\hat{E}(r, t) = \sqrt{2}\, \mathscr{E}_0 (\hat{a} e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t})\, u(r), \tag{2.4}$$

where $u(r)$ describes the spatial field distribution and is same in both classical and quantum theories. The constant $\mathscr{E}_0 = (\hbar\omega / 2\varepsilon_0 V)^{1/2}$ is equal to the "electric field per photon" in the cavity of volume V.

A particularly useful set of states is the overcomplete set of coherent states $|\alpha\rangle$ which are the eigenstates of the annihilation operator $\hat{a}$

$$\hat{a}\, |\alpha\rangle = \alpha\, |\alpha\rangle. \tag{2.5}$$

These coherent states can be generated from the vacuum state $|0\rangle$ [defined as $\hat{a}\, |0\rangle = 0$] by the action of the unitary displacement operator $\hat{D}(\alpha)$ [6]

$$\hat{D}(\alpha) \equiv \exp[\alpha \hat{a}^\dagger - \alpha^* \hat{a}]; \qquad |\alpha\rangle = \hat{D}(\alpha)\, |0\rangle. \tag{2.6}$$

The parametric space of eigenvalues, i.e., the *phase space* for our dynamical system, is the *infinite* plane of eigenvalues (q, p) of the Hermitean operators $\hat{q}$ and $\hat{p}$. An equivalent phase space is the complex plane of eigenvalues

$$\alpha = \frac{1}{\sqrt{2\hbar}} (\lambda q + i\lambda^{-1} p); \tag{2.7}$$

of the annihilation operator $\hat{a}$. We should note here that the coherent state $|\alpha\rangle$ is not an eigenstate of either $\hat{q}$ or $\hat{p}$. The quantities q and p in Eq. (2.7) can be interpreted as the expectation values of the operators $\hat{q}$ and $\hat{p}$ in the state $|\alpha\rangle$. Two invariant differential elements of the two phase-spaces are related as:

$$\frac{1}{\pi} d^2\alpha = \frac{1}{\pi} d[\mathrm{Re}(\alpha)]\, d[\mathrm{Im}(\alpha)] = \frac{1}{2\pi\hbar} dq\, dp. \tag{2.8}$$

The phase-space description of the quantum-mechanical oscillator which is in the state described by the density operator $\hat{\rho} = |\Psi\rangle\langle\Psi|$ (in what follows we will consider mainly pure states but the formalism presented here can be applied for statistical mixtures as well) is based on the definition of the Wigner function [4]

$W_{|\Psi\rangle}(\xi)$. The Wigner function is related to the characteristic function $C^{(W)}_{|\Psi\rangle}(\eta)$ of the Weyl-ordered moments of the annihilation and creation operators of the harmonic oscillator as follows [7]

$$W_{|\Psi\rangle}(\xi)=\frac{1}{\pi}\int C^{(W)}_{|\Psi\rangle}(\eta)\exp(\xi\eta^*-\xi^*\eta)\,d^2\eta. \tag{2.9}$$

The characteristic function $C^{(W)}_{|\Psi\rangle}(\eta)$ of the system described by the density operator $\hat{\rho}$ is defined as

$$C^{(W)}_{|\Psi\rangle}(\eta)\equiv\mathrm{Tr}[\hat{\rho}\hat{D}(\eta)], \tag{2.10}$$

where $\hat{D}(\eta)$ is the displacement operator given by Eq. (2.6). The characteristic function $C^{(W)}_{|\Psi\rangle}(\eta)$ can be used for the evaluation of the Weyl-ordered products of the annihilation and creation operators:

$$\langle\{(\hat{a}^\dagger)^m\,\hat{a}^n\}\rangle=\frac{\partial^{(m+n)}}{\partial\eta^m\partial(-\eta^*)^n}\,C^{(W)}_{|\Psi\rangle}(\eta)\bigg|_{\eta=0}. \tag{2.11}$$

On the other hand the mean value of the Weyl-ordered product $\langle\{(\hat{a}^\dagger)^m\,\hat{a}^n\}\rangle$ can be obtained by using the Wigner function directly:

$$\langle\Psi|\,\{(\hat{a}^\dagger)^m\,\hat{a}^n\}\,|\Psi\rangle=\frac{1}{\pi}\int d^2\xi(\xi^*)^m\,\xi^n W_{|\Psi\rangle}(\xi). \tag{2.12}$$

For instance, the Weyl-ordered product $\langle\{\hat{a}^\dagger\hat{a}^2\}\rangle$ can be evaluated as

$$\langle\{\hat{a}^\dagger\hat{a}^2\}\rangle=\frac{1}{3}\langle\hat{a}^\dagger\hat{a}^2+\hat{a}\hat{a}^\dagger\hat{a}+\hat{a}^2\hat{a}^\dagger\rangle=\frac{1}{\pi}\int d^2\xi\,|\xi|^2\,\xi W_{|\Psi\rangle}(\xi). \tag{2.13}$$

In this paper we will several times refer to mean values of central moments and cumulants of the system operators $\hat{a}$ and $\hat{a}^\dagger$. We will denote central moments as $\langle\cdots\rangle^{(c)}$ and in what follows we will consider the Weyl-ordered central moments which are defined as

$$\langle\{(\hat{a}^\dagger)^m\,\hat{a}^n\}\rangle^{(c)}\equiv\langle\{(\hat{a}^\dagger-\langle\hat{a}^\dagger\rangle)^m\,(\hat{a}-\langle\hat{a}\rangle)^n\}\rangle. \tag{2.14}$$

From this definition it follows that the central moments of the order k ($k=m+n$) can be expressed by moments of the order less or equal to k. On the other hand we denote cumulants as $\langle\!\langle\cdots\rangle\!\rangle$. The cumulants are usually defined via characteristic functions. In particular, the Weyl-ordered cumulants are defined as

$$\langle\!\langle\{(\hat{a}^\dagger)^m\,\hat{a}^n\}\rangle\!\rangle=\frac{\partial^{(m+n)}}{\partial\eta^m\partial(-\eta^*)^n}\ln C^{(W)}_{|\Psi\rangle}(\eta)\bigg|_{\eta=0}, \tag{2.15}$$

where $C^{(W)}_{|\Psi\rangle}(\eta)$ is the characteristic function of the Weyl-ordered moments given by Eq. (2.10). The cumulants of the order k ($k=m+n$) can be expressed in terms of moments of the order less or equal to k.

Originally the Wigner function was introduced in a form different from (2.9). Namely, the Wigner function was defined as a particular Fourier transform of the density operator expressed in the basis of the eigenvectors $|q\rangle$ of the position operator $\hat{q}$

$$W_{|\Psi\rangle}(q,p)\equiv\int_{-\infty}^{\infty} d\zeta\,\langle q-\zeta/2\,|\hat{\rho}|\,q+\zeta/2\rangle\,e^{ip\zeta/\hbar}, \tag{2.16a}$$

which for a pure state described by a state vector $|\Psi\rangle$ (i.e., $\hat{\rho}=|\Psi\rangle\langle\Psi|$) reads

$$W_{|\Psi\rangle}(q,p)\equiv\int_{-\infty}^{\infty} d\zeta\,\psi(q-\zeta/2)\,\psi^*(q+\zeta/2)\,e^{ip\zeta/\hbar}, \tag{2.16b}$$

where $\psi(q)\equiv\langle q\mid\Psi\rangle$. It can be shown that both definitions (2.9) and (2.16) of the Wigner function are identical (see Hillery *et al.* [4]), providing the parameters ξ and ξ^* are related to the coordinates q and p of the phase space as

$$\xi=\frac{1}{\sqrt{2\hbar}}(\lambda q+i\lambda^{-1}p);\qquad \xi^*=\frac{1}{\sqrt{2\hbar}}(\lambda q-i\lambda^{-1}p), \tag{2.17}$$

i.e.,

$$W_{|\Psi\rangle}(q,p)=\frac{1}{2\pi\hbar}\int C^{(W)}_{|\Psi\rangle}(q',p')\exp\left[-\frac{i}{\hbar}(qp'-pq')\right]dq'\,dp', \tag{2.18a}$$

where the characteristic function $C^{(W)}_{|\Psi\rangle}(q,p)$ is given by the relation

$$C^{(W)}_{|\Psi\rangle}(q,p)=\mathrm{Tr}[\hat{\rho}\hat{D}(q,p)]. \tag{2.18b}$$

The displacement operator in terms of the position and the momentum operators reads

$$\hat{D}(q,p)=\exp\left[\frac{i}{\hbar}(\hat{q}p-\hat{p}q)\right]. \tag{2.19}$$

The symmetrically ordered cumulants of the operators $\hat{q}$ and $\hat{p}$ can be evaluated as

$$\langle\!\langle\{\hat{p}^m\hat{q}^n\}\rangle\!\rangle=\hbar^{n+m}\frac{\partial^{(m+n)}}{\partial(-iq)^m\,\partial(ip)^n}\ln C^{(W)}_{|\Psi\rangle}(q,p)\bigg|_{q,p=0}. \tag{2.20}$$

The Wigner function can be interpreted as the quasiprobability (see below) density distribution through which a probability can be expressed to find a quantum-mechanical system (harmonic oscillator) around the "point" (q,p) of the phase space.

With the help of the Wigner function $W_{|\Psi\rangle}(q,p)$ the position and momentum probability distributions $W_{|\Psi\rangle}(q)$ and $W_{|\Psi\rangle}(p)$ can be expressed from $W_{|\Psi\rangle}(q,p)$ via marginal integration over the conjugated variable (in what follows we assume $\lambda = 1$)

$$W_{|\Psi\rangle}(q) \equiv \frac{1}{\sqrt{2\pi\hbar}} \int dp\, W_{|\Psi\rangle}(q,p) = \sqrt{2\pi\hbar}\, \langle q|\, \hat{\rho}\, |q\rangle, \tag{2.21a}$$

where $|q\rangle$ is the eigenstate of the position operator $\hat{q}$. The marginal probability distribution $W_{|\Psi\rangle}(q)$ is normalized to unity, i.e.,

$$\frac{1}{\sqrt{2\pi\hbar}} \int dq\, W_{|\Psi\rangle}(q) = 1. \tag{2.21b}$$

The relation (2.21a) for the probability distribution $W_{|\Psi\rangle}(q)$ of the position operator $\hat{q}$ can be generalized to the case of the distribution of the rotated quadrature operator $\hat{x}_\theta$. This operator is defined as

$$\hat{x}_\theta = \sqrt{\frac{\hbar}{2}}\, [\hat{a} e^{-i\theta} + \hat{a}^\dagger e^{i\theta}], \tag{2.22a}$$

and the corresponding conjugated operator $\hat{x}_{\theta+\pi/2}$, such that $[\hat{x}_\theta,\ \hat{x}_{\theta+\pi/2}] = i\hbar$, reads

$$\hat{x}_{\theta+\pi/2} = \frac{\sqrt{\hbar}}{i\sqrt{2}}\, [\hat{a} e^{-i\theta} - \hat{a}^\dagger e^{i\theta}]. \tag{2.22b}$$

The position and the momentum operators are related to the operator $\hat{x}_\theta$ as, $\hat{q} = \hat{x}_0$ and $\hat{x}_{\pi/2} = \hat{p}$. The rotation (i.e., the linear homogeneous canonical transformation) given by Eqs. (2.22) can be performed by the unitary operator $\hat{U}(\theta)$:

$$\hat{U}(\theta) = \exp[-i\theta \hat{a}^\dagger \hat{a}], \tag{2.23}$$

so that

$$\hat{x}_\theta = \hat{U}^\dagger(\theta)\, \hat{x}_0\, \hat{U}(\theta); \qquad \hat{x}_{\theta+\pi/2} = \hat{U}^\dagger(\theta)\, \hat{x}_{\pi/2}\, \hat{U}(\theta). \tag{2.24}$$

Alternatively, in the vector formalism we can rewrite the transformation (2.24) as

$$\begin{pmatrix} \hat{x}_\theta \\ \hat{x}_{\theta+\pi/2} \end{pmatrix} = \mathbf{F} \begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix}; \qquad \mathbf{F} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{2.25}$$

Eigenvalues x_θ and $x_{\theta+\pi/2}$ of the operators $\hat{x}_\theta$ and $\hat{x}_{\theta+\pi/2}$ can be expressed in terms of the eigenvalues q and p of the position and momentum operators as

$$\begin{pmatrix} x_\theta \\ x_{\theta+\pi/2} \end{pmatrix} = \mathbf{F} \begin{pmatrix} q \\ p \end{pmatrix}; \qquad \begin{pmatrix} q \\ p \end{pmatrix} = \mathbf{F}^{-1} \begin{pmatrix} x_\theta \\ x_{\theta+\pi/2} \end{pmatrix}; \qquad \mathbf{F}^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{2.26}$$

where the matrix $\mathbf{F}$ is given by Eq. (2.25) and $\mathbf{F}^{-1}$ is the corresponding inverse matrix. It has been shown by Ekert and Knight [10] that Wigner functions are transformed under the action of the linear canonical transformation (2.25) as

$$\begin{aligned} W_{|\Psi\rangle}(q, p) &\to W_{|\Psi\rangle}(\mathbf{F}^{-1}(x_\theta, x_{\theta+\pi/2})) \\ &= W_{|\Psi\rangle}(x_\theta \cos\theta - x_{\theta+\pi/2} \sin\theta;\, x_\theta \sin\theta + x_{\theta+\pi/2} \cos\theta), \end{aligned} \tag{2.27}$$

which means that the probability distribution $W_{|\Psi\rangle}(x_\theta) = \sqrt{2\pi\hbar}\, \langle x_\theta | \hat{\rho} | x_\theta \rangle$ can be evaluated as

$$W_{|\Psi\rangle}(x_\theta) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dx_{\theta+\pi/2}\, W_{|\Psi\rangle}(x_\theta \cos\theta - x_{\theta+\pi/2} \sin\theta;\, x_\theta \sin\theta + x_{\theta+\pi/2} \cos\theta). \tag{2.28}$$

As shown by Vogel and Risken [16] the knowledge of $W_{|\Psi\rangle}(x_\theta)$ for all values of θ (such that $[0 < \theta \leqslant \pi]$) is equivalent to the knowledge of the Wigner function itself. This Wigner function can be obtained from the set of distributions $W_{|\Psi\rangle}(x_\theta)$ via the inverse Radon transformation

$$\begin{aligned} W_{|\Psi\rangle}(q, p) = \frac{1}{(2\pi\hbar)^{3/2}} \int_{-\infty}^{\infty} dx_\theta \int_{-\infty}^{\infty} d\xi\, |\xi| \int_0^{\pi} d\theta\, W_{|\Psi\rangle}(x_\theta) \\ \times \exp\left[\frac{i}{\hbar}\xi(x_\theta - q\cos\theta - p\sin\theta)\right]. \end{aligned} \tag{2.29}$$

It will be shown later in this paper that the optical homodyne tomography is implicitly based on a measurement of all (in principle, infinite number) independent moments (cumulants) of the system operators. Nevertheless, there are states for which the Wigner function can be reconstructed much easier than via the homodyne tomography. These are Gaussian and generalized Gaussian states which are completely characterized by the first two cumulants of the relevant observables while all higher-order cumulants are equal to zero. On the other hand, if the state under consideration is characterized by an infinite number of nonzero cumulants then the homodyne tomography can fail because it does not provide us with a consistent truncation scheme (see below and [26]).

2.1. *States of Light to Be Considered*

In this paper we will consider several quantum-mechanical states of a single-mode light field. In particular, we will analyze coherent state, Fock state, squeezed vacuum state, and superpositions of coherent states.

A. *Coherent state.* The coherent state $|\alpha\rangle$ [see Eqs. (2.5-6)] is an eigenstate of the annihilation operator $\hat{a}$, i.e., $|\alpha\rangle$ is not an eigenstate of an observable. The

Wigner function [Eq. (2.9)] of the coherent state in the complex ξ-phase space reads

$$W_{|\alpha\rangle}(\xi) = 2\exp\left(-2\,|\xi-\alpha|^2\right); \qquad \alpha = \alpha_x + i\alpha_y, \tag{2.30a}$$

or alternatively, in the (q, p) phase space we have

$$W_{|\alpha\rangle}(q, p) = \frac{1}{\sigma_q \sigma_p} \exp\left[-\frac{1}{2\hbar}\frac{(q-\bar{q})^2}{\sigma_q^2} - \frac{1}{2\hbar}\frac{(p-\bar{p})^2}{\sigma_q^2}\right], \tag{2.30b}$$

where $\bar{q} = \sqrt{2\hbar}\,\alpha_x/\lambda$; $\bar{p} = \sqrt{2\hbar}\,\alpha_y\lambda$, and

$$\sigma_q^2 = \frac{1}{2\lambda^2} \qquad \text{and} \qquad \sigma_p^2 = \frac{\lambda^2}{2}. \tag{2.30c}$$

The mean photon number in the coherent state is equal to $\bar{n} = |\alpha|^2$. The variances for the position and momentum operators are

$$\langle\alpha|\,(\Delta\hat{q})^2\,|\alpha\rangle = \hbar\sigma_q^2; \qquad \langle\alpha|\,(\Delta\hat{p})^2\,|\alpha\rangle = \hbar\sigma_p^2, \tag{2.31}$$

from which it is seen that the coherent state belongs to the class of the minimum uncertainty states for which

$$\langle(\Delta\hat{q})^2\rangle\langle(\Delta\hat{p})^2\rangle = \hbar^2\sigma_q^2\sigma_p^2 = \frac{\hbar^2}{4}. \tag{2.32}$$

Using the expression (2.30b) for the Wigner function in the (q, p)-phase space we can evaluate the central moments of the Weyl-ordered moments of the operators $\hat{q}$ and $\hat{p}$ in the coherent state as

$$\langle\{\hat{q}^k\hat{p}^l\}\rangle^{(c)} = \begin{cases} (2n-1)!!\,(2m-1)!!\,(\hbar\sigma_q)^n\,(\hbar\sigma_p)^m; & \text{for} \quad k=2n,\, l=2m \\ 0; & \text{for} \quad k=2n+1 \text{ or } l=2m+1. \end{cases} \tag{2.33}$$

We see that all central moments of the order higher than second can be expressed in terms of the second-order central moments, so we can conclude that the coherent state is completely characterized by four mean values $\langle\hat{q}\rangle$; $\langle\hat{p}\rangle$; $\langle\hat{q}^2\rangle$, and $\langle\hat{p}^2\rangle$. With the help of the relation (2.18b) we can find the characteristic function $C_{|\alpha\rangle}^{(W)}(q, p)$ of the symmetrically-ordered moments of the coherent state

$$C_{|\alpha\rangle}^{(W)}(q, p) = \exp\left[\frac{i}{\hbar}\bar{q}p - \frac{i}{\hbar}\bar{p}q - \frac{\sigma_q^2}{2\hbar}p^2 - \frac{\sigma_p^2}{2\hbar}q^2\right], \tag{2.34}$$

from which the nonzero cumulants for the coherent state,

$$\langle\!\langle\hat{q}\rangle\!\rangle = \bar{q}; \qquad \langle\!\langle\hat{p}\rangle\!\rangle = \bar{p}; \qquad \langle\!\langle\hat{q}^2\rangle\!\rangle = \hbar\sigma_q^2; \qquad \langle\!\langle\hat{p}^2\rangle\!\rangle = \hbar\sigma_p^2, \tag{2.35}$$

can be found. We stress that all other cumulants of the operators $\hat{q}$ and $\hat{p}$ are equal to zero. This is due to the fact that the characteristic function of the Weyl-ordered moments is an exponential of a polynomial of the second order in q and p (for more discussion see Section 6).

B. *Fock State.* Eigenstates $|n\rangle$ of the photon number operator $\hat{n}$,

$$\hat{n} = \hat{a}^\dagger \hat{a} = \frac{1}{2\hbar}(\hat{q}^2 + \hat{p}^2) - \frac{1}{2}, \tag{2.36}$$

are called the Fock states. The Wigner function of the Fock state $|n\rangle$ is the ξ-phase space reads

$$W_{|n\rangle}(\xi) = 2(-1)^n \exp(-2|\xi|^2)\, \mathscr{L}_n(4|\xi|^2), \tag{2.37a}$$

where $\mathscr{L}_n(x)$ is the Laguerre polynomial of the order n. In the (q, p) phase space this Wigner function has the form

$$W_{|n\rangle}(q, p) = 2(-1)^n \exp\left(-\frac{q^2+p^2}{\hbar}\right) \mathscr{L}_n\left(2\,\frac{q^2+p^2}{\hbar}\right). \tag{2.37b}$$

The Wigner function (2.37b) does not have a Gaussian form. One can find from Eq. (2.37b) the following expressions for first few moments of the position and momentum operators

$$\begin{aligned}
\langle \hat{q} \rangle &= \langle \hat{p} \rangle = 0;\\
\langle \hat{q}^2 \rangle &= \langle \hat{p}^2 \rangle = \frac{\hbar}{2}(2n+1);\\
\langle \hat{q}^4 \rangle &= \langle \hat{p}^4 \rangle = \frac{\hbar^2}{4}(6n^2+6n+3) = \frac{3}{2}\,\frac{\langle \hat{q}^2\rangle^2 + \langle \hat{p}^2\rangle^2}{2} + \frac{3}{8}\hbar^2;\\
\langle \hat{q}^2\hat{p}^2 \rangle &= \langle \hat{p}^2\hat{q}^2 \rangle = \frac{\hbar^2}{4}(2n^2+2n-1) = \frac{1}{2}\,\frac{\langle \hat{q}^2\rangle^2 + \langle \hat{p}^2\rangle^2}{2} - \frac{3}{8}\hbar^2.
\end{aligned} \tag{2.38}$$

In addition we find for the characteristic function $C^{(W)}_{|n\rangle}(q, p)$ of the Weyl-ordered moments of the operators $\hat{q}$ and $\hat{p}$ in the Fock state $|n\rangle$ the expression

$$C^{(W)}_{|n\rangle}(q, p) = \exp\left[-\frac{(q^2+p^2)}{4\hbar}\right] \mathscr{L}_n\left(\frac{(q^2+p^2)}{2\hbar}\right), \tag{2.39}$$

from which it follows that the Fock state is characterized by an infinite number of nonzero cumulants. On the other hand, moments of the photon number operator $\hat{n}$ in the Fock state $|n\rangle$ are

$$\langle \hat{n}^k \rangle = n^k, \tag{2.40}$$

from which it follows higher-order moments of the operator $\hat{n}$ can be expressed in terms of the first-order moment and that all central moments $\langle \hat{n}^k \rangle^{(c)}$ are equal to zero.

C. *Squeezed vacuum state.* The squeezed vacuum state [27] can be expressed in the Fock basis as

$$|\eta\rangle = (1-\eta^2)^{1/4} \sum_{n=0}^{\infty} \frac{[(2n)!]^{1/2}}{2^n n!} \eta^n \, |2n\rangle, \tag{2.41a}$$

where the squeezing parameter η (for simplicity we assume η to be real) ranges from -1 to $+1$. The squeezed vacuum state (2.41a) can be obtained by the action of the squeezing operator $\hat{S}(r)$ on the vacuum state $|0\rangle$

$$|\eta\rangle = \hat{S}(r)\,|0\rangle; \qquad \hat{S}(r) = \exp\left[-\frac{ir}{2\hbar}(\hat{q}\hat{p} + \hat{p}\hat{q})\right] = \exp\left[\frac{r}{2}(\hat{a}^{\dagger 2} - \hat{a}^2)\right], \tag{2.41b}$$

where the squeezing parameter $r \in (-\infty, +\infty)$ is related to the parameter η as follows, $\eta = \tanh r$. The mean photon number in the squeezed vacuum (2.41) is given by the relation

$$\bar{n} = \frac{\eta^2}{1-\eta^2}. \tag{2.42}$$

The variances of the position and momentum operators can be expressed in a form (2.31) with the parameters σ_q and σ_p given by the relations

$$\sigma_q^2 = \frac{1}{2}\left(\frac{1+\eta}{1-\eta}\right); \qquad \sigma_p^2 = \frac{1}{2}\left(\frac{1-\eta}{1+\eta}\right). \tag{2.43}$$

If we assume the squeezing parameter to be real and $\eta \in [0, -1)$ then from Eq. (2.43) it follows that fluctuations in the momentum are reduced below the vacuum state limit $\hbar/2$ at the expense of increased fluctuations in the position. Simultaneously it is important to stress that the product of variances $\langle (\Delta\hat{q})^2 \rangle$ and $\langle (\Delta\hat{p})^2 \rangle$ is equal to $\hbar^2/4$, which means that the squeezed vacuum state belongs to the class of the minimum uncertainty states.

The Wigner function of the squeezed vacuum state is of the Gaussian form

$$W_{|\eta\rangle}(q, p) = \frac{1}{\sigma_q \sigma_p} \exp\left[-\frac{1}{2\hbar}\frac{q^2}{\sigma_q^2} - \frac{1}{2\hbar}\frac{p^2}{\sigma_p^2}\right], \tag{2.44}$$

with the parameters σ_q^2 and σ_p^2 given by Eq. (2.43). From Eq. (2.44) it follows that the mean value of the position and the momentum operators in the squeezed vacuum state are equal to zero, while the higher-order symmetrically-ordered (central) moments are given by Eq. (2.33) with the parameters σ_q^2 and σ_p^2 given by Eq. (2.43). We see that higher-order moments can be expressed in terms of the second-order moments. We can find the expression for the characteristic function $C^{(W)}_{|\eta\rangle}(q,p)$ for the squeezed vacuum state which reads

$$C^{(W)}_{|\eta\rangle}(q,p) = \exp\left[-\frac{\sigma_q^2}{2h}p^2 - \frac{\sigma_p^2}{2h}q^2\right], \tag{2.45}$$

from which it directly follows that the squeezed vacuum state is completely characterized by two nonzero cumulants $\langle\!\langle \hat{q}^2 \rangle\!\rangle = \hbar\sigma_q^2$ and $\langle\!\langle \hat{p}^2 \rangle\!\rangle = \hbar\sigma_p^2$ (all other cumulants are equal to zero).

D. *Even and odd coherent states.* In nonlinear optical processes superpositions of coherent states can be produced [28]. In particular, Brune et al. [29] have shown that an atomic-phase detection quantum non-demolition scheme can serve for production of superpositions of two coherent states of a single-mode radiation field. The superpositions

$$|\alpha_e\rangle = N_e^{1/2}(|\alpha\rangle + |-\alpha\rangle); \qquad N_e^{-1} = 2\,[1 + \exp(-2\,|\alpha|^2)], \tag{2.46a}$$

and

$$|\alpha_o\rangle = N_o^{1/2}(|\alpha\rangle - |-\alpha\rangle); \qquad N_o^{-1} = 2[1 - \exp(-2\,|\alpha|^2)], \tag{2.46b}$$

which are called the even and odd coherent states, respectively, can be produced via this scheme. These states have been introduced by Dodonov et al. [30] in a formal group-theoretical analysis of various subsystems of coherent states. More recently, these states have been analyzed as prototypes of superposition states of light [28] which exhibit various nonclassical effects. In particular, quantum interference between component states leads to oscillations in the photon number distributions. Another consequence of this interference is a reduction (squeezing) of quadrature fluctuations in the even coherent state. On the other hand, the odd coherent state exhibits reduced fluctuations in the photon number distribution (sub-Poissonian photon statistics). Nonclassical effects associated with superposition states can be explained in terms of quantum interference between the "points" (coherent states) in phase space. The character of quantum interference is very sensitive with respect to the relative phase between coherent components of superposition states. To illustrate this effect we write down the expressions for the Wigner functions of the even and odd coherent states (in what follows we assume α to be real)

$$W_{|\alpha_e\rangle}(q,p) = N_e[\,W_{|\alpha\rangle}(q,p) + W_{|-\alpha\rangle}(q,p) + W_{\mathrm{int}}(q,p)]; \tag{2.47a}$$

$$W_{|\alpha_o\rangle}(q,p) = N_o[\,W_{|\alpha\rangle}(q,p) + W_{|-\alpha\rangle}(q,p) - W_{\mathrm{int}}(q,p)], \tag{2.47b}$$

where the Wigner functions $W_{|\pm\alpha\rangle}(q,p)$ of coherent states $|\pm\alpha\rangle$ are given by Eq. (2.30b). The interference part of the Wigner functions (2.47) is given by the relation

$$W_{\mathrm{int}}(q,p) = \frac{2}{\sigma_q\sigma_p}\exp\left[-\frac{q^2}{2\hbar\sigma_q^2} - \frac{p^2}{2\hbar\sigma_p^2}\right]\cos\left(\frac{\bar{q}p}{\hbar\sigma_q\sigma_p}\right), \tag{2.48}$$

where $\bar{q} = \sqrt{2\hbar}\,\alpha$ (we assume real α) and the variances σ_q^2 and σ_p^2 are given by Eq. (2.30c). From Eqs. (2.47) it follows that the even and odd coherent states differ by a sign of the interference part, which results in completely different quantum-statistical properties of these states.

With the help of the Wigner function (2.47a) we evaluate mean values of moments of the operators $\hat{q}$ and $\hat{p}$. The first moments are equal to zero, i.e., $\langle \hat{q} \rangle = \langle \hat{p} \rangle = 0$, while for higher-order moments we find

$$
\begin{aligned}
\langle \hat{q}^2 \rangle &= \frac{\hbar}{2}(1 + 8N_e\alpha^2);\\
\langle \hat{p}^2 \rangle &= \frac{\hbar}{2}(1 - 8N_e\alpha^2 e^{-2\alpha^2});\\
\langle \hat{q}^4 \rangle &= \frac{3\hbar^2}{4}\left[1 + 16N_e\alpha^2\left(1 + \frac{2}{3}\alpha^2\right)\right];\\
\langle \hat{p}^4 \rangle &= \frac{3\hbar^2}{4}\left[1 - 16N_e\alpha^2 e^{-2\alpha^2}\left(1 - \frac{2}{3}\alpha^2\right)\right].
\end{aligned}
\tag{2.49}
$$

From Eqs. (2.49) it follows that the even coherent state exhibits the second and fourth-order squeezing in the $\hat{p}$-quadrature [28]. We do not present explicit expression for higher-order moments, which in general cannot be expressed in powers of second-order moments. In terms of the cumulants it means that the even (and odd) coherent states are characterized by an infinite number of nonzero cumulants. This can be seen from the expression for the characteristic function of the even coherent state which reads

$$
C^{(W)}_{|\alpha_e\rangle}(q,p) = 2N_e \exp\left[-\frac{\sigma_p^2}{2\hbar}q^2 - \frac{\sigma_q^2}{2\hbar}p^2\right]\left\{\cos\left(\frac{\bar{q}p}{\hbar}\right) + \exp\left(-\frac{\bar{q}^2}{2\hbar\sigma_q^2}\right)\cosh\left(\frac{\sigma_p}{\hbar\sigma_q}\bar{q}q\right)\right\}.
\tag{2.50}
$$

3. MaxEnt Principle and Observation Levels

The state of a quantum system can always be described by a statistical density operator $\hat{\rho}$. Depending on the system preparation, the density operator represents either a pure quantum state (complete system preparation) or a statistical mixture of pure states (incomplete preparation). The degree of deviation of a statistical mixture from the pure state can be best described by the *uncertainty measure* $\eta[\hat{\rho}]$ (see [22, 25])

$$
\eta[\hat{\rho}] = -k_B \operatorname{Tr}(\hat{\rho}\ln\hat{\rho}),
\tag{3.1}
$$

where k_B is the Boltzmann constant. The uncertainty measure $\eta[\hat{\rho}]$ possesses the following properties:

1. In the eigenrepresentation of the density operator $\hat{\rho}$

$$\hat{\rho}\,|r_m\rangle = r_m\,|r_m\rangle, \tag{3.2}$$

we can write

$$\eta[\hat{\rho}] = -k_B \sum_m r_m \ln r_m \geqslant 0, \tag{3.3}$$

where r_m are eigenvalues and $|r_m\rangle$ the eigenstates of $\hat{\rho}$.

2. For uncertainty measure $\eta[\hat{\rho}]$ the inequality

$$0 \leqslant \eta[\hat{\rho}] \leqslant k_{\mathrm{B}} \ln N \tag{3.4}$$

holds, where N denotes the dimension of the Hilbert space of the system and $\eta[\hat{\rho}]$ takes its maximum value when

$$\hat{\rho} = \frac{\hat{1}}{\mathrm{Tr}\,\hat{1}} = \frac{\hat{1}}{N}. \tag{3.5}$$

In this case all pure states in the mixture appear with the same probability equal to $1/N$. If the system is prepared in a pure state then it holds that $\eta[\hat{\rho}] = 0$.

3. It can be shown with the help of the Liouville equation

$$\frac{\partial}{\partial t}\hat{\rho}(t) = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}(t)], \tag{3.6}$$

that in the case of an isolated system the uncertainty measure is a constant of motion, i.e.,

$$\frac{d\eta(t)}{dt} = 0. \tag{3.7}$$

3.1. *MaxEnt Principle*

When instead of the density operator $\hat{\rho}$, expectation values G_ν of a set $\mathcal{O}$ of operators $\hat{G}_\nu$ $(\nu = 1, ..., n)$ are given, then the uncertainty measure can be determined as well. The set of linearly independent operators is referred to as the *observation level* $\mathcal{O}$ [22]. The operators $\hat{G}_\nu$ which belong to a given observation level do not commutate necessarily. A large number of density operators which fulfill the conditions

$$\mathrm{Tr}\,\hat{\rho}_{\{\hat{G}\}} = 1, \tag{3.8a}$$

$$\mathrm{Tr}(\hat{\rho}_{\{\hat{G}\}}\hat{G}_\nu) = G_\nu, \qquad \nu = 1, 2, ..., n; \tag{3.8b}$$

can be found for a given set of expectation values $G_\nu = \langle \hat{G}_\nu \rangle$. Each of these density operators $\hat{\rho}_{\{\hat{G}\}}$ can posses a different value of the uncertainty measure $\eta[\hat{\rho}_{\{\hat{G}\}}]$. If we wish to use only the expectation values G_ν of the chosen observation level for determining the density operator, we must select a particular density operator $\hat{\rho}_{\{\hat{G}\}} = \hat{\sigma}_{\{\hat{G}\}}$ in an unbiased manner. According to the Jaynes principle of the Maximum Entropy [24] this density operator $\hat{\sigma}_{\{\hat{G}\}}$ must be the one which has the largest uncertainty measure

$$\eta_{\max} \equiv \max\{\eta[\hat{\rho}_{\{\hat{G}\}}]\} = \eta[\hat{\sigma}_{\{\hat{G}\}}] \tag{3.9}$$

and simultaneously fulfills constraints (3.8). As a consequence of Eq.(3.9) the fundamental inequality

$$\eta[\hat{\sigma}_{\{\hat{G}\}}] = -k_{\mathrm{B}} \operatorname{Tr}(\hat{\sigma}_{\{\hat{G}\}} \ln \hat{\sigma}_{\{\hat{G}\}}) \geqslant \eta[\hat{\rho}_{\{\hat{G}\}}] = -k_{\mathrm{B}} \operatorname{Tr}(\hat{\rho}_{\{\hat{G}\}} \ln \hat{\rho}_{\{\hat{G}\}}) \tag{3.10}$$

holds for all possible $\hat{\rho}_{\{\hat{G}\}}$ which fulfill Eqs. (3.8). The variation determining the maximum of $\eta[\hat{\rho}_{\{\hat{G}\}}]$ under the conditions (3.8) leads to a generalized canonical density operator [23, 24, 31]

$$\hat{\sigma}_{\{\hat{G}\}} = \frac{1}{Z_{\{\hat{G}\}}} \exp\left(-\sum_\nu \lambda_\nu \hat{G}_\nu\right), \tag{3.11}$$

$$Z_{\{\hat{G}\}}(\lambda_1, ..., \lambda_n) = \operatorname{Tr}\left[\exp\left(-\sum_\nu \lambda_\nu \hat{G}_\nu\right)\right], \tag{3.12}$$

where λ_n are the Lagrange multipliers and $Z_{\{\hat{G}\}}(\lambda_1, ...\lambda_n)$ is the generalized partition function. By using the derivatives of the partition function we obtain the expectation values G_ν as

$$G_\nu = \operatorname{Tr}(\hat{\sigma}_{\{\hat{G}\}} \hat{G}_\nu) = -\frac{\partial}{\partial \lambda_\nu} \ln Z_{\{\hat{G}\}}(\lambda_1, ..., \lambda_n), \tag{3.13}$$

where in the case of noncommuting operators the following relation has to be used

$$\frac{\partial}{\partial a} \exp[-\hat{X}(a)] = -\exp[-\hat{X}(a)] \int_0^1 \exp[\mu \hat{X}(a)] \frac{\partial \hat{X}(a)}{\partial a} \exp[-\mu \hat{X}(a)]\, d\mu. \tag{3.14}$$

By using Eq. (3.13) the Lagrange multipliers can, in principle, be expressed as functions of the expectation values

$$\lambda_\nu = \lambda_\nu(G_1, ..., G_n). \tag{3.15}$$

We note that Eqs. (3.13) for Langrange multipliers not always have solutions which lead to physical results (see Section 6.2), which means that in these cases states of quantum systems cannot be reconstructed on a given observation level.

The maximum uncertainty measure regarding an observation level $\mathcal{O}_{\{\hat{G}\}}$ will be referred to as the entropy $S_{\{\hat{G}\}}$:

$$S_{\{\hat{G}\}} \equiv \eta_{\max} = -k_B \operatorname{Tr}(\hat{\sigma}_{\{\hat{G}\}} \ln \hat{\sigma}_{\{\hat{G}\}}). \tag{3.16}$$

This means that to different observation levels different entropies are related. By inserting $\sigma_{\{\hat{G}\}}$ [cf. Eq. (3.11)] into Eq. (3.16), we obtain the expression for the entropy

$$S_{\{\hat{G}\}} = k_B \ln Z_{\{\hat{G}\}} + k_B \sum_{\nu} \lambda_\nu G_\nu. \tag{3.17}$$

By making use of Eq. (3.15), the parameters λ_ν in the above equation can be expressed as functions of the expectation values G_ν and this leads to a new expression for the entropy

$$S_{\{\hat{G}\}} = S(G_1, ..., G_n). \tag{3.18}$$

We note that using the expression

$$dS_{\{\hat{G}\}} = k_B \sum_{\nu} \lambda_\nu \, dG_\nu \tag{3.19}$$

which follows from Eqs. (3.13) and (3.17) the following relation can be obtained:

$$k_{\mathrm{B}} \lambda_\nu = \frac{\partial}{\partial G_\nu} S(G_1, ..., G_n). \tag{3.20}$$

3.2. *Linear Transformations within an Observation Level*

An observation level can be defined either by a set of linearly independent operators $\{\hat{G}_\nu\}$, or by a set of independent linear combinations of the same operators

$$\hat{G}'_\mu = \sum_{\nu} c_{\mu\nu} \hat{G}_\nu. \tag{3.21}$$

Therefore, $\hat{\sigma}$ and S are invariant under a linear transformation

$$\hat{\sigma}'_{\{\hat{G}'\}} = \frac{\exp(-\sum_\mu \lambda'_\mu \hat{G}'_\mu)}{\operatorname{Tr} \exp(-\sum_\mu \lambda'_\mu \hat{G}'_\mu)} = \hat{\sigma}_{\{\hat{G}\}}. \tag{3.22}$$

As a result, the Lagrange multipliers transform contravariantly to Eq. (3.21), i.e.,

$$\lambda'_\mu = \sum_{\nu} c'_{\mu\nu} \lambda_\nu, \tag{3.23}$$

$$\sum_{\mu} c'_{\nu\mu} c_{\mu\rho} = \delta_{\nu\rho}. \tag{3.24}$$

3.3. *Extension and Reduction of the Observation Level*

If an observation level $\mathcal{O}_{\{\hat{G}\}} \equiv \hat{G}_1, ..., \hat{G}_n$ is extended by including further operators $\hat{M}_1, ..., \hat{M}_l$, then additional expectation values $M_1 = \langle \hat{M}_1 \rangle, ..., M_l = \langle \hat{M}_l \rangle$ can only increase amount of available information about the state of the system. This procedure is called the *extension* of the observation level (from $\mathcal{O}_{\{\hat{G}\}}$ to $\mathcal{O}_{\{\hat{G}, \hat{M}\}}$) and is associated with a decrease of the entropy. More precisely, the entropy $S_{\{\hat{G}, \hat{M}\}}$ of the extended observation level $\mathcal{O}_{\{\hat{G}, \hat{M}\}}$ can be only smaller or equal to the entropy $S_{\{\hat{G}\}}$ of the original observation level $\mathcal{O}_{\{\hat{G}\}}$,

$$S_{\{\hat{G}, \hat{M}\}} \leqslant S_{\{\hat{G}\}}. \tag{3.25}$$

The generalized canonical density operator of the observation level $\mathcal{O}_{\{\hat{G}, \hat{M}\}}$

$$\hat{\sigma}_{\{\hat{G}, \hat{M}\}} = \frac{1}{Z_{\{\hat{G}, \hat{M}\}}} \exp\left(- \sum_{\nu=1}^{n} \lambda_\nu \hat{G}_\nu - \sum_{\mu=1}^{l} \kappa_\mu \hat{M}_\mu \right), \tag{3.26a}$$

with

$$Z_{\{\hat{G}, \hat{M}\}} = \mathrm{Tr}\left[\exp\left(- \sum_{\nu=1}^{n} \lambda_\nu \hat{G}_\nu - \sum_{\mu=1}^{l} \kappa_\mu \hat{M}_\mu \right) \right], \tag{3.26b}$$

belongs to the set of density operators $\hat{\rho}_{\{\hat{G}\}}$ fulfilling Eq. (3.8). Therefore, Eq. (3.25) is a special case of Eq. (3.11). Analogously to Eqs. (3.13) and (3.15), the Lagrange multipliers can be expressed by functions of the expectation values

$$\lambda_\nu = \lambda_\nu(G_1, ..., G_n, M_1, ..., M_l), \tag{3.27a}$$

$$\kappa_\mu = \kappa_\mu(G_1, ..., G_n, M_1, ..., M_l). \tag{3.27b}$$

The sign of equality in Eq. (3.25) holds only for $\kappa_\mu = 0$. In this special case the expectation values M_μ are functions of the expectation values G_ν, and the operators $\hat{M}_\mu$ can be expressed as functions of $\hat{G}_\nu$. The measurement of observables $\hat{M}_\mu$ does not increase information about the system. Consequently, $\hat{\rho}_{\{\hat{G}, \hat{M}\}} = \hat{\rho}_{\{\hat{G}\}}$ and $S_{\{\hat{G}, \hat{M}\}} = S_{\{\hat{G}\}}$.

We can also consider a *reduction of the observation level* if we decrease number of independent observables which are measured, e.g., $\mathcal{O}_{\{\hat{G}, \hat{M}\}} \to \mathcal{O}_{\{\hat{G}\}}$ (here $\hat{G}_\nu$ and $\hat{M}_\mu$ are independent). This reduction is accompanied with an increase of the entropy due to the decrease of the information available about the system.

3.4. *Time-Dependent Entropy of an Observation Level*

If the dynamical evolution of the system is governed by the evolution superoperator $\hat{U}(t, t_0)$, such that $\hat{\rho}(t) = \hat{U}(t, t_0)\, \hat{\rho}(t_0)$, then expectation values of the operators $\hat{G}_\nu$ on the given observation level at time t read

$$G_\nu(t) = \mathrm{Tr}[\hat{G}_\nu \hat{U}(t, t_0)\, \hat{\rho}(t_0)]. \tag{3.28}$$

By using these time-dependent expectation values as constraints for maximizing the uncertainty measure $\eta[\hat{\rho}_{\{\hat{G}\}}(t)]$, we get the generalized canonical density operator

$$\hat{\sigma}_{\{\hat{G}\}}(t) = \frac{\exp(-\sum_\nu \lambda_\nu(t)\,\hat{G}_\nu)}{\mathrm{Tr}[\exp(-\sum_\nu \lambda_\nu(t)\,\hat{G}_\nu)]} \tag{3.29}$$

and the time-dependent entropy of the corresponding observation level

$$S_{\{\hat{G}\}}(t) = -k_\mathrm{B}\,\mathrm{Tr}[\hat{\sigma}_{\{\hat{G}\}}(t)\ln\hat{\sigma}_{\{\hat{G}\}}(t)] = k_\mathrm{B}\ln Z_{\{\hat{G}\}}(t) + k_B\sum_\nu \lambda_\nu(t)\,G_\nu(t). \tag{3.30}$$

This generalized canonical density operator does not satisfy the von Neumann equation but it satisfies an integro-differential equation derived by Robertson and Seke [23, 31]. The time-dependent entropy is defined for any system being arbitrarily far from equilibrium. In the case of an isolated system the entropy can increase or decrease during the time evolution (see, for example Ref. [25], Sec. 5.6).

3.5. *Wigner Functions on Different Observation Levels*

With the help of a generalized canonical density operator $\hat{\sigma}_{\{\hat{G}\}}$ we define the Wigner function in the ξ phase space at the corresponding observation level as

$$W_{\{\hat{G}\}}(\xi) = \frac{1}{\pi}\int d^2\eta\,\mathrm{Tr}[\hat{D}(\eta)\,\hat{\sigma}_{\{\hat{G}\}}]\exp(\xi\eta^* - \xi^*\eta). \tag{3.31}$$

Analogous expression can be found for the Wigner function in the (q, p) phase space [see Eq. (2.18)].

3.6. *MaxEnt Principle and Laws of Physics*

It has been pointed out by Jaynes in his Brandeis lectures [24] that there is a strong formal resemblance between the MaxEnt formalism and the rules of calculations in statistical mechanics and thermodynamics. Simultaneously he has emphasized that the MaxEnt principle "has nothing to do with the laws of physics."[2] To be more specific it is worth to cite a paragraph from the Jaynes' Brandeis lectures (see p. 183 of these lectures [24]): "Conventional quantum theory has provided an answer to the problem of setting up initial state descriptions only in the limiting case where measurements of a "complete set of commuting observables" have been made, the density matrix $\hat{\rho}(0)$ then reducing to the projection operator onto a pure state $\psi(0)$ which is the appropriate simultaneous eigenstate of all measured quantities. But there is almost no experimental situation in which we really have all this information, and before we have a theory able to treat actual experimental situations, existing quantum theory must be supplemented with some principle that tells us how to translate, or encode, the results of measurements into a definite state

[2] In fact, this is the reason why the MaxEnt principle is applicable in so many fields of human activities, for instance we can mention economy or sociology (for more details see the book by Kapur and Kesavan [25]).

description $\hat{\rho}(0)$. Note that the problem is not to find $\hat{\rho}(0)$ which correctly describes "true physical situation". That is unknown, and always remains so, because of incomplete information. In order to have a usable theory we must ask the much more modest question: *What* $\hat{\rho}(0)$ *best describes our state of knowledge about the physical situation*?".

In other words, the MaxEnt principle is *the most conservative assignment in the sense that it does not permit one to draw any conclusions not warranted by the data.* From this point of view the MaxEnt principle has a very close relation (or can be understood as the generalization) of the Laplace's principle of *indifference* [32] which states that where nothing is known one should choose a constant valued function to reflect this ignorance. Then it is just a question how to quantify a degree of this ignorance. If we choose an entropy to quantify the ignorance, then the relation between the Laplace's indifference principle and the Jaynes principle of the Maximum Entropy is transparent, i.e. for a constant-valued probability distribution the entropy takes its maximum value.

We can conclude that a measurement itself is a physical process and is governed by the laws of physics. On the other hand formal procedures by means of which we reconstruct information about the system from the measured data are based on certain principles which cannot be directly expressed in terms of the physical laws. From this point of view the MaxEnt principle which is used in the present paper has close relations to the reconstruction procedure proposed recently by Jones [33] which is based on the Shannon information theory and the Bayesian theory for inverting quantum data.

4. Observation Levels for Single-Mode Field

In our paper we will consider two different classes of observation levels. Namely, we will consider the phase-sensitive and phase-insensitive observation levels. These two classes do differ by the fact that phase-sensitive observation levels are related to such operators which provide some information about off-diagonal matrix elements of the density operator in the Fock basis (i.e., these observation levels reveal some information about the phase of states under consideration). On the contrary, phase-insensitive observation levels are based exclusively on a measurement of diagonal matrix elements in the Fock basis. Before we proceed to a detailed description of the phase-sensitive and phase-insensitive observation levels we introduce two exceptional observation levels, the complete and thermal observation levels.

Complete Observation Level $\mathcal{O}_0 \equiv \{(\hat{a}^\dagger)^k\, \hat{a}^l;\, \forall k, l\}$

The set of operators $|n\rangle\langle m|$ (for all n and m) is referred to as *complete* observation level. Expectation values of the operators $|n\rangle\langle m|$ are the matrix elements of the density operator in the Fock basis

$$\langle m|\, \hat{\rho}\, |n\rangle = \mathrm{Tr}[\,\hat{\rho}\, |n\rangle\langle m|\,]; \qquad \forall n, m, \tag{4.1}$$

and therefore the generalized canonical density operator is identical with the statistical density operator

$$\hat{\sigma}_0 = \frac{1}{Z_0} \exp\left[-\sum_{m,n=0}^{\infty} \lambda_{m,n} |n\rangle\langle m| \right] = \hat{\rho}; \tag{4.2a}$$

$$Z_0 = \mathrm{Tr}\left\{ \exp\left[-\sum_{m,n=0}^{\infty} \lambda_{m,n} |n\rangle\langle m| \right] \right\}. \tag{4.2b}$$

In this case the entropy S_0 is determined by the density operator $\hat{\rho}$ as

$$S_0 = -k_B \,\mathrm{Tr}[\hat{\sigma}_0 \ln \hat{\sigma}_0] = -k_B \,\mathrm{Tr}[\hat{\rho} \ln \hat{\rho}]. \tag{4.3}$$

This entropy is usually called the von Neumann entropy [13].

As a consequence of the relation (cf. Sec. 3.3 in [34])

$$|n\rangle\langle m| = \lim_{\varepsilon \to 1} \sum_{k=0}^{\infty} \frac{(-\varepsilon)^k}{k! \sqrt{n!\, m!}} (\hat{a}^\dagger)^{k+n} \hat{a}^{k+m}, \tag{4.4}$$

the complete observation level $\mathcal{O}_0$ can also be given by a set of operators $\{(\hat{a}^\dagger)^k \hat{a}^l;\ \forall k, l\}$ or $\{\hat{q}^k \hat{p}^l;\ \forall k, l\}$. The Wigner function on the complete information level is equal to the Wigner function of the state itself, i.e., $W^{(0)}_{|\Psi\rangle}(\xi) = W_{|\Psi\rangle}(\xi)$.

Thermal Observation Level $\mathcal{O}_{\mathrm{th}} \equiv \{\hat{a}^\dagger \hat{a}\}$.

The total reduction of the complete observation level $\mathcal{O}_0$ results in a thermal observation level $\mathcal{O}_{\mathrm{th}}$ characterized just by one observable, the photon number operator $\hat{n}$, i.e., quantum-mechanical states of light on this observation level are characterized only by their mean photon number $\bar{n} \equiv \langle \hat{n} \rangle$. The generalized canonical density operator of this observation level is the well-known density operator of the harmonic oscillator in the thermal equilibrium

$$\hat{\sigma}_{\mathrm{th}} = \frac{1}{Z_{\mathrm{th}}} \exp[-\lambda_{\mathrm{th}} \hat{n}]. \tag{4.5}$$

To find an explicit expression for the Lagrange multiplier λ_{th} we have to solve the equation

$$\mathrm{Tr}[\sigma_{\mathrm{th}} \hat{n}] = \bar{n}, \tag{4.6a}$$

from which we find that

$$\lambda_{\mathrm{th}} = \ln\left(\frac{\bar{n}+1}{\bar{n}} \right), \tag{4.6b}$$

so that the partition function corresponding to the operator $\hat{\sigma}_{\text{th}}$ reads

$$Z_{\text{th}} = \{1 - \exp[-\lambda_{\text{th}}]\}^{-1} = \bar{n} + 1. \tag{4.7}$$

Now we can rewrite the generalized canonical density operator $\hat{\sigma}_{\text{th}}$ in the Fock basis in a form

$$\hat{\sigma}_{\text{th}} = \sum_{n=0}^{\infty} \frac{\bar{n}^n}{(\bar{n}+1)^{n+1}} |n\rangle\langle n|. \tag{4.8}$$

For the entropy of the thermal observation level we find a familiar expression

$$S_{\text{th}} = k_{\text{B}}(\bar{n}+1) \ln(\bar{n}+1) - k_B \bar{n} \ln \bar{n}\,. \tag{4.9}$$

The fact that the entropy S_{th} is larger than zero for any $\bar{n} > 0$ reflects the fact that on the thermal observation level *all* states with the same mean photon number are indistinguishable. This is the reason why Wigner function of different states on the thermal information level are identical. The Wigner function of the state $|\Psi\rangle$ on the thermal observation level is given by the relation

$$W^{(\text{th})}_{|\Psi\rangle}(\xi) = \frac{2}{1+2\bar{n}} \exp\left[-\frac{2|\xi|^2}{1+2\bar{n}}\right]. \tag{4.10}$$

Extending the thermal observation level we can obtain more "realistic" Wigner functions which in the limit of the complete observation level are equal to the Wigner function of the measured state itself, i.e., they are not biased by the lack of information (measured data) about the state.

4.1. *Phase-Sensitive Observation Levels*

4.1.1. *Observation level* $\mathcal{O}_1 \equiv \{\hat{a}^\dagger\hat{a}, \hat{a}^\dagger, \hat{a}\}$. We can extent the thermal observation level if in addition to the observable $\hat{n}$ we consider also the measurement of mean values of the operators $\hat{a}$ and $\hat{a}^\dagger$ (that is, we consider a measurement of the observables $\hat{q}$ and $\hat{p}$). If we denote the (measured) mean values of this operators as $\langle\hat{a}\rangle = \gamma$ and $\langle\hat{a}^\dagger\rangle = \gamma^*$, then the generalized canonical density operator $\hat{\sigma}_1$ can be written as

$$\hat{\sigma}_1 = \frac{1}{Z_1} \exp[-\lambda_1(\hat{a}^\dagger - \gamma^*)(\hat{a} - \gamma)], \tag{4.11a}$$

with the partition function Z_1 given by the relation

$$Z_1 = (1 - e^{-\lambda_1})^{-1}. \tag{4.11b}$$

We have chosen the density operator $\hat{\sigma}_1$ in such form that the conditions

$$\langle\hat{a}\rangle = \text{Tr}[\hat{a}\hat{\sigma}_1] = \gamma; \qquad \langle\hat{a}^\dagger\rangle = \text{Tr}[\hat{a}^\dagger\hat{\sigma}_1] = \gamma^*, \tag{4.12}$$

are automatically fulfilled. To see this we rewrite the density operator $\hat{\sigma}_1$ in the form

$$\hat{\sigma}_1 = \frac{1}{Z_1} \hat{D}(\gamma) \exp[-\lambda_1 \hat{a}^\dagger \hat{a}]\, \hat{D}^\dagger(\gamma), \tag{4.13a}$$

where we have used the transformation property $\hat{D}(\gamma)\, \hat{a}\hat{D}^\dagger(\gamma) = \hat{a} - \gamma$, and therefore

$$\begin{aligned} \mathrm{Tr}[\hat{a}\hat{\sigma}_1] &= \frac{1}{Z_1} \mathrm{Tr}[\hat{D}^\dagger(\gamma)\, \hat{a}\hat{D}(\gamma) \exp(-\lambda_1 \hat{a}^\dagger \hat{a})] \\ &= \gamma + \frac{1}{Z_1} \mathrm{Tr}[\hat{a} \exp(-\lambda_1 \hat{a}^\dagger \hat{a})] = \gamma. \end{aligned} \tag{4.13b}$$

To find the Lagrange multiplier λ_1 we have to solve the equation $\mathrm{Tr}[\hat{a}^\dagger \hat{a}\hat{\sigma}_1] = \bar{n}$ from which we find

$$e^{-\lambda_1} = \frac{\bar{n} - |\gamma|^2}{1 + \bar{n} - |\gamma|^2}. \tag{4.14}$$

The entropy S_1 on the observation level $\mathcal{O}_1$ can be expressed in a form very similar to S_{th} [see Eq. (4.9)]

$$S_1 = k_B[\bar{n} - |\gamma|^2 + 1] \ln [\bar{n} - |\gamma|^2 + 1] - k_B[\bar{n} - |\gamma|^2] \ln [\bar{n} - |\gamma|^2]. \tag{4.15}$$

The Wigner function $W^{(1)}_{|\Psi\rangle}(\xi)$ corresponding to the generalized canonical density operator $\hat{\sigma}_1$ reads

$$W^{(1)}_{|\Psi\rangle}(\xi) = \frac{2}{1 + 2(\bar{n} - |\gamma|^2)} \exp\left[-\frac{2\, |\xi - \gamma|^2}{1 + 2(\bar{n} - |\gamma|^2)} \right]. \tag{4.16}$$

From the expression (4.15) for the entropy S_1 it follows that $S_1 = 0$ for those states for which $\bar{n} = |\gamma|^2$. In fact, there is only one state with this property. It is a coherent state $|\alpha\rangle$ (2.6). In other words, because of the fact that $S_1 = 0$, the coherent state can be *completely* reconstructed on the observation level $\mathcal{O}_1$. In this case

$$W^{(1)}_{|\alpha\rangle}(\xi) = W^{(0)}_{|\alpha\rangle}(\xi) = 2 \exp[-2\, |\xi - \alpha|^2], \tag{4.17}$$

[see Eq. (2.30)]. For other states $S_1 > 0$ and therefore to improve our information about the state we have to perform further measurements, i.e., we have to extent the observation level $\mathcal{O}_1$.

4.1.2. *Observation level* $\mathcal{O}_2 \equiv \{\hat{a}^\dagger \hat{a}, (\hat{a}^\dagger)^2, \hat{a}^2, \hat{a}^\dagger, \hat{a}\}$. One of possible extensions of the observation level $\mathcal{O}_1$ can be performed with a help of observables $\hat{q}^2$ and $\hat{p}^2$, i.e., when not only the mean photon number $\bar{n}$ and mean values of $\hat{q}$ and $\hat{p}$ are known, but also the variances $\langle(\Delta\hat{q})^2\rangle$, $\langle(\Delta\hat{p})^2\rangle$, and $\langle\{\Delta\hat{q}\,\Delta\hat{p}\}\rangle$ are measured. On the observation level $\mathcal{O}_2$ we can express the generalized canonical operator $\hat{\sigma}_2$ as

$$\hat{\sigma}_2 = \frac{1}{Z_2}\exp\left[-\frac{\lambda_2}{2}(\hat{a}^\dagger-\gamma^*)^2 - \frac{\lambda_2^*}{2}(\hat{a}-\gamma)^2 - \lambda_1(\hat{a}^\dagger-\gamma^*)(\hat{a}-\gamma)\right], \tag{4.18}$$

where the Lagrange multiplier λ_1 is real while λ_2 can be complex: $\lambda_2 = |\lambda_2|\,e^{-i\theta}$. We can rewrite $\hat{\sigma}_2$ in a form similar to the thermal density operator

$$\hat{\sigma}_2 = \frac{1}{\tilde{Z}_2}\hat{D}(\gamma)\,\hat{U}(\theta/2)\,\hat{S}(r)\exp[-(\lambda_1^2-|\lambda_2|^2)^{1/2}\,\hat{a}^\dagger\hat{a}]\,\hat{S}^\dagger(r)\,\hat{U}^\dagger(\theta/2)\,\hat{D}^\dagger(\gamma), \tag{4.19a}$$

where the operators $\hat{D}(\gamma)$, $\hat{U}(\theta/2)$, and $\hat{S}(r)$ are given by Eqs. (2.6), (2.23), and (2.41b), respectively. These operators transform the annihilation operator $\hat{a}$ as

$$\begin{aligned}\hat{D}^\dagger(\gamma)\,\hat{a}\hat{D}(\gamma) &= \hat{a}+\gamma;\\ \hat{U}^\dagger(\theta/2)\,\hat{a}\hat{U}(\theta/2) &= \hat{a}e^{-i\theta/2};\\ \hat{S}^\dagger(r)\,\hat{a}\hat{S}(r) &= \hat{a}\cosh r + \hat{a}^\dagger\sinh r.\end{aligned} \tag{4.19b}$$

The partition function $\tilde{Z}_2$ in Eq. (4.19a) can be evaluated in an explicit form

$$\tilde{Z}_2^{-1} = 1-\exp[-(\lambda_1^2-|\lambda_2|^2)^{1/2}]. \tag{4.19c}$$

In Eq. (4.19a) we have chosen the parameter r to be given by the relation $\tanh 2r = -|\lambda_2|/\lambda_1$. The density operator (4.19a) is defined in such way that it automatically fulfills the condition $\mathrm{Tr}[\hat{a}\hat{\sigma}_2]=\gamma$, while the Lagrange multipliers λ_1 and λ_2 have to be found from the relations $\mathrm{Tr}[\hat{a}^\dagger\hat{a}\hat{\sigma}_2]=\bar{n}$ and $\mathrm{Tr}[\hat{a}^2\hat{\sigma}_2]=\mu$:

$$\begin{aligned}\mathrm{Tr}[\hat{a}^\dagger\hat{a}\hat{\sigma}_2] &= \bar{n} = |\gamma|^2 - 1/2 + (\chi+1/2)\cosh 2r;\\ \mathrm{Tr}[\hat{a}^2\hat{\sigma}_2] &= \mu = \gamma^2 + e^{-i\theta}(\chi+1/2)\sinh 2r,\end{aligned} \tag{4.20a}$$

where we have used the notation

$$\chi = \{\exp[(\lambda_1^2-|\lambda_2|^2)^{1/2}]-1\}^{-1}. \tag{4.20b}$$

Instead of finding explicit expressions for the Lagrange multipliers λ_1 and λ_2 we can find solutions for the parameters $\tanh 2r$ and χ. We express these parameters in terms of the measured central moments $\langle\hat{a}^\dagger\hat{a}\rangle^{(c)} \equiv N = \bar{n}-|\gamma|^2 > 0$ and $\langle\hat{a}^2\rangle^{(c)} \equiv M = |M|\,e^{-i\theta} = \mu-\gamma^2$:

$$\tanh 2r = \frac{|M|}{N+1/2}, \tag{4.21a}$$

$$\chi = [(N+1/2)^2-|M|^2]^{1/2}-1/2. \tag{4.21b}$$

We remind us that physical requirements [35] lead to the following restrictions on the parameters N and M:

$$N \geqslant 0; \qquad N(N+1) \geqslant |M|^2. \tag{4.22}$$

Once the $\tanh 2r$ and χ are found we can reconstruct the Wigner function $W^{(2)}_{|\Psi\rangle}(\xi)$ on the observation level $\mathcal{O}_2$. This Wigner function reads

$$W^{(2)}_{|\Psi\rangle}(\xi) = \frac{1}{[(N+1/2)^2 - |M|^2]^{1/2}} \times \exp\left[-\frac{(N+1/2)\,|\xi-\gamma|^2 - (M^*/2)(\xi-\gamma)^2 - (M/2)(\xi^*-\gamma^*)^2}{[(N+1/2)^2 - |M|^2]} \right]. \tag{4.23}$$

Analogously we can find an expression for the entropy S_2:

$$S_2 = k_{\mathrm{B}}(\chi+1)\ln(\chi+1) - k_{\mathrm{B}}\chi \ln \chi. \tag{4.24}$$

It has a form of the thermal entropy (4.9) with a mean thermal-photon number equal to χ [see Eq. (4.21b)].

Using the expression for the Wigner function (4.23) we can rewrite the variances of the position and momentum operators in terms of the parameters N and M as

$$\langle (\Delta\hat{q})^2 \rangle = \frac{\hbar}{2}[1 + 2N + 2\mathrm{Re}\, M]; \qquad \langle (\Delta\hat{p})^2 \rangle = \frac{\hbar}{2}[1 + 2N - 2\,\mathrm{Re}\, M]. \tag{4.25}$$

The product of these variances reads

$$\langle (\Delta\hat{q})^2 \rangle \langle (\Delta\hat{p})^2 \rangle = \frac{\hbar^2}{4}[(1+2N)^2 - 4(\mathrm{Re}\, M)^2]. \tag{4.26a}$$

From the expression (4.24) for the entropy S_2 it is seen that those states for which $N(N+1) = |M|^2$ can be completely reconstructed of the observation level $\mathcal{O}_2$, because for these states $S_2 = 0$. In fact, it has been shown by Dodonov *et al.* [36] that the states for which $N(N+1) = |M|^2$ are the *only* pure states which have non-negative Wigner functions [see Eq. (4.21)]. For these states the product of variances (4.26a) reads

$$\langle (\Delta\hat{q})^2 \rangle \langle (\Delta\hat{p})^2 \rangle = \frac{\hbar^2}{4}[1 + 4(\mathrm{Im}\, M)^2], \tag{4.26b}$$

which means that if in addition $\mathrm{Im}\, M = 0$ (see for instance squeezed vacuum state with real parameter of squeezing) then these states also belong to the class of the minimum uncertainty states. From our previous discussion it follows that the squeezed vacuum as well as squeezed coherent states can be completely reconstructed on the observation level $\mathcal{O}_2$. More generally, we can say that all pure Gaussian states for which $N(N+1)=|M|^2$ can be completely reconstructed on this observation level.

4.1.3. *Higher-order phase-sensitive observation levels.* There are pure non-Gaussian states (such as the even coherent state) for which the entropy S_2 is larger than zero and therefore in order to reconstruct Wigner functions of such states more precisely, we have to extent the observation level $\mathcal{O}_2$. Straightforward extension of $\mathcal{O}_2$ is the observation level $\mathcal{O}_k \equiv \{(\hat{a}^\dagger)^m \hat{a}^n;\ \forall m, n; m+n \leqslant k\}$, which in the limit $k \to \infty$ is extended to the complete observation level.

To perform a reconstruction of the Wigner function on the observation level $\mathcal{O}_k$ with $k>2$ an attention has to be paid to the fact that for a certain choice of possible observables the vacuum-to-vacuum matrix elements of the generalized canonical density operator $\langle 0|\, \hat{\sigma}_k\, |0\rangle$ can have divergent Taylor-series expansion. To be more specific, if we consider an observation level such that $\mathcal{O}_k \equiv \{(\hat{a}^\dagger)^k,\ \hat{a}^k\}$ then for the generalized canonical density operator

$$\hat{\sigma}_k = \frac{1}{Z_k} \exp[-\lambda_k (\hat{a}^\dagger)^k - \lambda_k^* \hat{a}^k], \tag{4.27}$$

the corresponding partition function $Z_k = \mathrm{Tr} \exp[-\lambda_k (\hat{a}^\dagger)^k - \lambda_k^* \hat{a}^k]$ is divergent [37]. This means that one cannot consistently define an observation level based exclusively on the measurement of the operators $(\hat{a}^\dagger)^k$ and $\hat{a}^k$. In general, to "regularize" the problem one has to include the photon number operator $\hat{n}$ into the observation level. Then the generalized density operator $\hat{\sigma}_k$,

$$\hat{\sigma}_k = \frac{1}{Z_k} \exp[-\lambda_0 \hat{a}^\dagger \hat{a} - \lambda_k (\hat{a}^\dagger)^k - \lambda_k^* \hat{a}^k], \tag{4.28}$$

can be properly defined and one may reconstruct the corresponding Wigner function $W_k(\xi)$. We note that any observation has to be chosen in such a way that information about the mean photon number is available, i.e., knowledge of the mean photon number (the mean energy) of the system under consideration represents a necessary condition for a reconstruction of the Wigner function (see also Appendix A).

4.2. *Phase-Insensitive Observation Levels*

The choice of the observation level is very important in order to optimize the strategy for the measurement and the reconstruction of the Wigner function of a given quantum-mechanical state of light. For instance, if we would like to reconstruct the Wigner function of the Fock state $|n\rangle$ at the observation level $\mathcal{O}_k \equiv \{\hat{a}^\dagger \hat{a},\ (\hat{a}^\dagger)^m \hat{a}^n; m+n \leqslant k \text{ and } m \neq n\}$ we find that irrespectively on the number (k) of "measured" moments $\langle (\hat{a}^\dagger)^m \hat{a}^n \rangle$ (for $m \neq n$) the reconstructed Wigner function is always equal to the thermal Wigner function (4.10). So it can happen that in a very tedious experiment negligible information is obtained. On the other hand, if a measurement of diagonal elements of the density operator in the Fock basis is performed relevant information can be obtained much easier.

4.2.1. *Observation level* $\mathcal{O}_A \equiv \{\hat{P}_n = |n\rangle\langle n|; \forall n\}$. The most general phase-insensitive observation level corresponds to the case when *all* diagonal elements $P_n = \langle n| \hat{\rho} |n\rangle$ of the density operator $\hat{\rho}$ describing the state under consideration are measured. The observation level $\mathcal{O}_A$ can be obtained via a reduction of the complete observation level $\mathcal{O}_0$ and it corresponds to the measurement of the photon number distribution P_n such that $\sum_n P_n = 1$. Because of the relation [see Eq. (4.4)]

$$|n\rangle\langle n| = \lim_{\varepsilon \to 1} \sum_{k=0}^{\infty} \frac{(-\varepsilon)^k}{k!n!} (\hat{a}^\dagger)^{k+n} \hat{a}^{k+n} = \lim_{\varepsilon \to 1} \sum_{k=0}^{\infty} \frac{(-\varepsilon)^k}{k!n!} \frac{\hat{n}!}{(\hat{n}-k-n)!}, \tag{4.29}$$

we can conclude that the observation level $\mathcal{O}_A$ corresponds to the measurement of all moments of the creation and annihilation operators of the form $(\hat{a}^\dagger)^k \hat{a}^k$ or, what is the same, it corresponds to a measurement of all moments of the photon number operator, i.e.,

$$\mathcal{O}_A \equiv \{\hat{P}_n = |n\rangle\langle n|; \forall n\} = \{(\hat{a}^\dagger)^k \hat{a}^k; \forall k\} = \{\hat{n}^k; \forall k\}. \tag{4.30}$$

The generalized canonical operator $\hat{\sigma}_A$ at the observation level $\mathcal{O}_A$ reads

$$\hat{\sigma}_A = \frac{1}{Z_A} \exp\left[-\sum_{n=0}^{\infty} \lambda_n |n\rangle\langle n| \right]; \tag{4.31a}$$

with the partition function given by the relation

$$Z_A = \mathrm{Tr}\left\{ \exp\left[-\sum_{n=0}^{\infty} \lambda_n |n\rangle\langle n| \right] \right\} = \sum_{n=0}^{\infty} \exp[-\lambda_n]. \tag{4.31b}$$

The entropy S_A at the observation level $\mathcal{O}_A$ can be expressed in the form

$$S_A = k_B \ln Z_A + k_B \sum_{n=0}^{\infty} \lambda_n P_n. \tag{4.32}$$

The Lagrange multipliers λ_n have to be evaluated from an infinite set of equations:

$$P_n = \mathrm{Tr}[\hat{\sigma}_A \hat{P}_n] = \frac{e^{-\lambda_n}}{Z_A}; \qquad \forall n, \tag{4.33}$$

from which we find $\lambda_n = -\ln[Z_A P_n]$. If we insert λ_n into expression (4.32) we obtain for the entropy S_A the familiar expression

$$S_A = -k_B \sum_{n=0}^{\infty} P_n \ln P_n, \tag{4.34}$$

derived by Shannon [38]. Here it should be briefly noted that as a consequence of the relation

$$\sum_{n=0}^{\infty} \hat{P}_n = \hat{1}, \tag{4.35}$$

the operators $\hat{P}_n$ are not linearly independent, which means that the Lagrange multipliers λ_n and the partition function Z_{A} are not uniquely defined. Nevertheless, if Z_{A} is chosen to be equal to unity, then the Lagrange multipliers can be expressed as

$$\lambda_n = -\ln P_n; \tag{4.36a}$$

and the generalized canonical density operator reads

$$\hat{\sigma}_{\text{A}} = \sum_{n=0}^{\infty} P_n \, |n\rangle\langle n|; \qquad \sum_{n=0}^{\infty} P_n = 1. \tag{4.36b}$$

From here it follows that the Wigner function $W^{(\text{A})}_{|\Psi\rangle}(\xi)$ of the state $|\Psi\rangle$ at the observation level $\mathcal{O}_{\text{A}}$ can be reconstructed in the form

$$W^{(\text{A})}_{|\Psi\rangle}(\xi) = \sum_{n=0}^{\infty} P_n W_{|n\rangle}(\xi), \tag{4.37}$$

where $W_{|n\rangle}(\xi)$ is the Wigner function of the Fock state $|n\rangle$ given by Eq. (2.37).

The phase-insensitive observation level $\mathcal{O}_{\text{A}}$ can be further reduced if only a finite number of operators $\hat{P}_n$ [where $n \in \mathcal{M}$] is considered. In this case, in general, we have $\sum_{n\in\mathcal{M}} P_n < 1$ and therefore it is essential that apart of mean values P_n also the mean photon number $\bar{n}$ is known from the measurement. In Appendix A we analyze the situation when the operator $\hat{n}$ is not included into the observation level. We show that in this case no reliable information about the system is obtained even though many P_n's are known (but $\sum_{n\in\mathcal{M}} P_n < 1$).

4.2.2. *Observation level* $\mathcal{O}_{\text{B}} \equiv \{\hat{n}, \hat{P}_{2n} = |2n\rangle\langle 2n|; \forall n\}$. As an example of the observation level which is reduced with respect to $\mathcal{O}_{\text{A}}$ we can consider the observation level $\mathcal{O}_{\text{B}}$ which is based on a measurement of the average photon number $\bar{n}$ and on the photon statistics on the subspace of the Fock space composed of the even Fock states $|2n\rangle$. In this case the generalized canonical density operator $\hat{\sigma}_{\text{B}}$ can be written as

$$\hat{\sigma}_{\text{B}} = \frac{1}{Z_{\text{B}}} \exp\left[-\lambda\hat{n} - \sum_{n=0}^{\infty} \lambda_n \hat{P}_{2n}\right] = \frac{e^{-\lambda\hat{n}}}{Z_{\text{B}}}\left[\left(1 - \sum_{n=0}^{\infty} \hat{P}_{2n}\right) + \sum_{n=0}^{\infty} e^{-\lambda_n} \hat{P}_{2n}\right], \tag{4.38a}$$

where the partition function is given by the relation

$$Z_{\text{B}} = \text{Tr}\left\{\exp\left[-\lambda\hat{n} - \sum_{n=0}^{\infty} \lambda_n \hat{P}_{2n}\right]\right\}. \tag{4.38b}$$

This partition function can be explicitly evaluated with the help of solutions for the Lagrange multipliers from equations $\mathrm{Tr}[\hat{P}_{2n}\hat{\sigma}_{\mathrm{B}}]=P_{2n}$. If we introduce the notation

$$P_{\mathrm{odd}} \equiv 1-\sum_{n=0}^{\infty} P_{2n}; \tag{4.39a}$$

$$\bar{n}_{\mathrm{odd}} \equiv \bar{n}-\sum_{n=0}^{\infty} 2nP_{2n}, \tag{4.39b}$$

then the partition function Z_{B} can be expressed as

$$Z_{\mathrm{B}}=\frac{[\bar{n}_{\mathrm{odd}}^2-P_{\mathrm{odd}}^2]^{1/2}}{2P_{\mathrm{odd}}^2}. \tag{4.40}$$

Analogously we find for the generalized canonical density operator the expression

$$\hat{\sigma}_{\mathrm{B}}=\sum_{n=0}^{\infty} P_{2n}\,|2n\rangle\langle 2n|+\sum_{n=0}^{\infty} P_{2n+1}\,|2n+1\rangle\langle 2n+1|, \tag{4.41}$$

where P_{2n} are measured values of $\hat{P}_{2n}$ and P_{2n+1} are evaluated from the MaxEnt principle:

$$P_{2n+1}=\frac{2P_{\mathrm{odd}}^2}{\bar{n}_{\mathrm{odd}}+P_{\mathrm{odd}}}\left(\frac{\bar{n}_{\mathrm{odd}}-P_{\mathrm{odd}}}{\bar{n}_{\mathrm{odd}}+P_{\mathrm{odd}}}\right)^n. \tag{4.42}$$

From Eq. (4.42) we see that on the subspace of odd Fock states we have obtained from the MaxEnt principle a "thermal-like" photon number distribution. Now, we know all values of P_{2n} and P_{2n+1} and using Eq. (4.34) we can easily evaluate the entropy S_{B} and the Wigner function $W^{(\mathrm{B})}_{|\Psi\rangle}(\xi)$ on the observation level $\mathcal{O}_{\mathrm{B}}$ [see Eq. (4.37)].

4.2.3. *Observation level* $\mathcal{O}_{\mathrm{C}} \equiv \{\hat{n}, \hat{P}_{2n+1}=|2n+1\rangle\langle 2n+1|; \forall n\}$. If the mean photon number and the probabilities $P_{2n+1}=\langle 2n+1|\,\hat{\rho}\,|2n+1\rangle$ are known, then we can define an observation level $\mathcal{O}_{\mathrm{C}}$ which in a sense is a complementary observation level to $\mathcal{O}_{\mathrm{B}}$. After some algebra one can find for the generalized canonical density operator $\hat{\sigma}_{\mathrm{C}}$ the expression equivalent to Eq. (4.41), i.e.,

$$\hat{\sigma}_{\mathrm{C}}=\sum_{n=0}^{\infty} P_{2n}\,|2n\rangle\langle 2n|+\sum_{n=0}^{\infty} P_{2n+1}\,|2n+1\rangle\langle 2n+1|, \tag{4.43}$$

where the parameters P_{2n+1} are known from measurement and P_{2n} are evaluated as

$$P_{2n}=\frac{2P_{\mathrm{even}}^2}{\bar{n}_{\mathrm{even}}+2P_{\mathrm{even}}}\left(\frac{\bar{n}_{\mathrm{even}}}{\bar{n}_{\mathrm{even}}+2P_{\mathrm{even}}}\right)^n. \tag{4.44}$$

In Eq. (4.44) we have introduced notations

$$P_{\text{even}} \equiv 1 - \sum_{n=0}^{\infty} P_{2n+1}; \tag{4.45a}$$

$$\bar{n}_{\text{even}} \equiv \bar{n} - \sum_{n=0}^{\infty} (2n+1)\, P_{2n+1}. \tag{4.45b}$$

The explicit expression for the partition function Z_C is

$$Z_C = \frac{\bar{n}_{\text{even}} + 2P_{\text{even}}}{2P_{\text{even}}^2}. \tag{4.46}$$

The reconstruction of the Wigner function $W_{|\Psi\rangle}^{(C)}(\xi)$ is now straightforward [see Eq. (4.37)].

4.2.4. *Observation level* $\mathcal{O}_D \equiv \{\hat{n}, \hat{P}_N = |N\rangle\langle N|\}$. We can reduce observation levels $\mathcal{O}_{A, B, C}$ even further and we can consider only a measurement of the mean photon number $\bar{n}$ and a probability P_N to find the system under consideration in the Fock state $|N\rangle$. The generalized density operator $\hat{\sigma}_D$ in this case reads

$$\hat{\sigma}_D = \frac{1}{Z_D} \exp[-\lambda \hat{n} - \lambda_N \hat{P}_N]. \tag{4.47}$$

Taking into account the fact that the observables under consideration do commute, i.e., $[\hat{n}, \hat{P}_N] = 0$, and that the operator $\hat{P}_N$ is a projector (i.e., $\hat{P}_N^2 = \hat{P}_N$) we can rewrite Eq. (4.47) as

$$\hat{\sigma}_D = \frac{e^{-\lambda \hat{n}}}{Z_D} [(1 - \hat{P}_N) + e^{-\lambda_N} \hat{P}_N] = P_N |N\rangle\langle N| + \sum_{n \neq N}^{\infty} P_n |n\rangle\langle n|, \tag{4.48}$$

where λ and λ_N are Lagrange multipliers associated with operators $\hat{n}$ and $\hat{P}_N$, respectively, and $P_n = \exp(-\lambda n)/Z_D$ gives the photon number distribution on the subspace of the Fock space without the vector $|N\rangle$. The generalized partition function can be expressed as

$$Z_D = \frac{1}{1-x} + x^N(y-1), \tag{4.49a}$$

where we have introduced notation

$$x = \exp(-\lambda); \qquad y = \exp(-\lambda_N). \tag{4.49b}$$

The Lagrange multipliers can be found from equations

$$P_N = \frac{1}{Z_{\mathrm{D}}} x^N y = \frac{(1-x)\, x^N y}{1 + x^N(y-1)(1-x)}; \tag{4.50a}$$

$$\bar{n} = \frac{1}{Z_{\mathrm{D}}} \left[\frac{x}{(1-x)^2} + N x^N (y-1) \right] = \frac{x + N x^N (1-x)^2 (y-1)}{(1-x)[1 + x^N(y-1)(1-x)]}. \tag{4.50b}$$

Generally, we cannot express the Lagrange multipliers λ and λ_N as functions of $\bar{n}$ and P_N in an analytical way for arbitrary N and Eqs. (4.50) have to be solved numerically. Nevertheless, there are two cases when these equations can be solved in a closed analytical form.

1. If $N=0$ (we will denote this observation level as $\mathcal{O}_{\mathrm{D1}}$), then we can find for Lagrange multipliers λ and λ_0 the expressions

$$e^{-\lambda} = 1 - \frac{1-P_0}{\bar{n}}; \qquad e^{-\lambda_0} = \frac{P_0}{(1-P_0)^2} [\bar{n} - (1-P_0)]; \tag{4.51a}$$

and for the partition function we find

$$Z_{\mathrm{D1}} = \frac{\bar{n} - (1-P_0)}{(1-P_0)^2}. \tag{4.51b}$$

Then after some straightforward algebra we can evaluate the parameters P_n as

$$P_n = \begin{cases} P_0 & \text{for} \quad n=0; \\ \dfrac{(1-P_0)^2}{\bar{n} - (1-P_0)} \left[\dfrac{\bar{n} - (1-P_0)}{\bar{n}} \right]^n & \text{for} \quad n>0. \end{cases} \tag{4.52}$$

From Eq. (4.52) which describes the photon number distribution obtained from the generalized density operator $\hat{\sigma}_{\mathrm{D1}}$ it follows that the reconstructed state on the observation level $\mathcal{O}_{\mathrm{D1}}$ has on the subspace form of Fock states except the vacuum a thermal-like character. Nevertheless, in this case the reconstructed Wigner function can be negative (unlike in the case of the thermal observation level). This can happen if P_0 is close to zero and $\bar{n}$ is small (see Section 5.1). Using explicit expressions for the parameters P_n given by Eq.(4.52) we can evaluate the entropy S_{D1} corresponding to the present observation level

$$S_{\mathrm{D1}} = -k_{\mathrm{B}} P_0 \ln P_0 - k_{\mathrm{B}}(\bar{n} - P) \ln (\bar{n} - P) - 2k_{\mathrm{B}} P \ln P + k_{\mathrm{B}} \bar{n} \ln \bar{n}, \tag{4.53}$$

where we have used notation $P = 1 - P_0$. In the limit $P_0 \to (1+\bar{n})^{-1}$ expression (4.53) reads

$$\lim_{P_0 \to (1+\bar{n})^{-1}} S_{\mathrm{D1}} = k_{\mathrm{B}}(\bar{n}+1) \ln (\bar{n}+1) - k_{\mathrm{B}} \bar{n} \ln \bar{n}, \tag{4.54}$$

which is the entropy on the thermal observation level Eq. (4.9). In this limit the $\mathcal{O}_{\mathrm{D1}}$ reduces to the thermal observation level $\mathcal{O}_{\mathrm{th}}$. On the other hand, in the limit $P_0 \to 0$ we obtain from Eq. (4.53)

$$\lim_{P_0 \to 0} S_{\mathrm{D1}} = k_{\mathrm{B}} \bar{n} \ln \bar{n} - k_{\mathrm{B}} (\bar{n} - 1) \ln (\bar{n} - 1), \tag{4.55}$$

from which it directly follows that in this case the mean photon number has necessary to be larger or equal than unity. Moreover, from Eq. (4.55) we see that in the limit $\bar{n} \to 1$ the entropy $S_{\mathrm{D1}} = 0$ which means that the Fock state $|1\rangle$ can be completely reconstructed on the observation level $\mathcal{O}_{\mathrm{D1}}$. This fact can also be seen from an explicit expression for the photon number distribution (4.52) from which it follows that

$$\lim_{\bar{n} \to 1} \lim_{P_0 \to 0} P_n = \delta_{n,1}. \tag{4.56}$$

2. If the mean photon number is an integer, then in the case $N = \bar{n}$ (we will denote this observation level as $\mathcal{O}_{\mathrm{D2}}$) we find for the Lagrange multipliers λ and $\lambda_{N=\bar{n}} \equiv \lambda_{\bar{n}}$ the expressions

$$e^{-\lambda} = \frac{\bar{n}}{1+\bar{n}}; \qquad e^{-\lambda_{\bar{n}}} = \frac{(1+\bar{n})^{1+\bar{n}} - \bar{n}^{\bar{n}}}{(1-P_{\bar{n}})\, \bar{n}^{\bar{n}}} P_{\bar{n}}, \tag{4.57a}$$

and for the partition function we find

$$Z_{\mathrm{D2}} = \frac{(1+\bar{n})^{1+\bar{n}} - \bar{n}^{\bar{n}}}{(1-P_{\bar{n}})(1+\bar{n})^{\bar{n}}}. \tag{4.57b}$$

Taking into account the expression for the reconstructed photon number distribution

$$P_n = \langle n | \hat{\sigma}_{\mathrm{D2}} | n \rangle = \frac{e^{-n\lambda}}{Z_{\mathrm{D2}}} [1 + \delta_{n,\bar{n}} (e^{-\lambda_{\bar{n}}} - 1)], \tag{4.58a}$$

then with the help of relations (4.57) we find

$$P_n = \begin{cases} P_{\bar{n}}; & n = \bar{n} \\ \dfrac{(1-P_{\bar{n}})(1+\bar{n})^{\bar{n}}}{(1+\bar{n})^{1+\bar{n}} - \bar{n}^{\bar{n}}} \left(\dfrac{\bar{n}}{1+\bar{n}} \right)^n; & n \neq \bar{n}. \end{cases} \tag{4.58b}$$

We see that the reconstructed photon-number distribution has a thermal-like character. The corresponding entropy can be evaluated in a closed analytical form

$$S_{\mathrm{D2}} = -k_{\mathrm{B}} P_{\bar{n}} \ln P_{\bar{n}} - k_{\mathrm{B}} (1-P_{\bar{n}}) \ln (1-P_{\bar{n}}) + k_B (1-P_{\bar{n}}) \ln \left[\frac{(1+\bar{n})^{1+\bar{n}}}{\bar{n}^{\bar{n}}} - 1 \right]. \tag{4.59}$$

It is interesting to note that if $P_{\bar{n}}$ is given by its value in the thermal photon number distribution, i.e.,

$$P_{\bar{n}} = \frac{\bar{n}^{\bar{n}}}{(1+\bar{n})^{1+\bar{n}}}, \tag{4.60a}$$

then the entropy (4.59) reduces to

$$S_{\mathrm{D2}} = k_{\mathrm{B}}(\bar{n}+1)\ln(\bar{n}+1) - k_{\mathrm{B}}\bar{n}\ln\bar{n} = -k_{\mathrm{B}}\ln P_{\bar{n}}, \tag{4.60b}$$

which means that the reconstructed density operator $\hat{\sigma}_{\mathrm{D2}}$ on the observation level $\mathcal{O}_{\mathrm{D2}}$ with $P_{\bar{n}}$ given by Eq. (4.60a) is equal to the density operator of the thermal field [see Eq. (4.8)] and so, in this case the reduction $\mathcal{O}_{\mathrm{D2}} \rightarrow \mathcal{O}_{\mathrm{th}}$ takes place. On the other hand, if $P_{\bar{n}} = 1$ then $S_{\mathrm{D2}} = 0$ and the Fock state $|\bar{n}\rangle$ can be completely reconstructed on the observation level $\mathcal{O}_{\mathrm{D2}}$.

4.3. *Relations between Observation Levels*

Various observation levels considered in this section can be obtained as a result of a sequence of mutual reductions. Therefore we can order observation levels under consideration. This ordering can be done separately for phase-sensitive and phase-insensitive observation levels. In particular, phase-sensitive observation levels are ordered:

$$\mathcal{O}_0 \supset \mathcal{O}_2 \supset \mathcal{O}_1 \supset \mathcal{O}_{\mathrm{th}}. \tag{4.61}$$

The corresponding entropies are related as

$$S_0 \leqslant S_2 \leqslant S_1 \leqslant S_{\mathrm{th}}. \tag{4.62}$$

The ordering of phase-insensitive observation levels $\mathcal{O}_{\mathrm{A}}$, $\mathcal{O}_{\mathrm{B}}$, $\mathcal{O}_{\mathrm{C}}$, $\mathcal{O}_{\mathrm{D1}}$ and $\mathcal{O}_{\mathrm{D2}}$ is more complex. In particular, we find

$$\mathcal{O}_0 \supset \mathcal{O}_{\mathrm{A}} \supset \left\{ \begin{matrix} \mathcal{O}_{\mathrm{B}} \\ \mathcal{O}_{\mathrm{C}} \end{matrix} \right\} \supset \mathcal{O}_{\mathrm{th}}; \tag{4.63a}$$

$$\mathcal{O}_0 \supset \mathcal{O}_{\mathrm{A}} \supset \left\{ \begin{matrix} \mathcal{O}_{\mathrm{D1}} \\ \mathcal{O}_{\mathrm{D2}} \end{matrix} \right\} \supset \mathcal{O}_{\mathrm{th}}, \tag{4.63b}$$

and

$$\mathcal{O}_0 \supset \mathcal{O}_{\mathrm{A}} \supset \mathcal{O}_{\mathrm{B}} \supset \mathcal{O}_{\mathrm{D1}} \supset \mathcal{O}_{\mathrm{th}} \tag{4.63c}$$,

which reflects the fact that observation levels $\mathcal{O}_{\mathrm{B}}$ and $\mathcal{O}_{\mathrm{C}}$ (as well as $\mathcal{O}_{\mathrm{D1}}$ and $\mathcal{O}_{\mathrm{D2}}$) cannot be obtained as a result of mutual reduction or extension. The corresponding entropies are related as

$$S_0 \leqslant S_{\rm A} \leqslant \begin{Bmatrix} S_{\rm B} \\ S_{\rm C} \end{Bmatrix} \leqslant S_{\rm th}, \tag{4.64a}$$

$$S_0 \leqslant S_{\rm A} \leqslant \begin{Bmatrix} S_{\rm D1} \\ S_{\rm D2} \end{Bmatrix} \leqslant S_{\rm th}, \tag{4.64b}$$

and

$$S_0 \leqslant S_{\rm A} \leqslant S_{\rm B} \leqslant S_{\rm D1} \leqslant S_{\rm th}. \tag{4.64c}$$

For a particular quantum-mechanical state of light observation levels $\mathcal{O}_{\rm X}$ can be ordered with respect to increasing values of entropies $S_{\rm X}$. From the above it also follows that if the entropy $S_{\rm X}$ on the observation level $\mathcal{O}_{\rm X}$ is equal to zero, then the entropies on the extended observation levels are equal to zero as well. It means that the complete reconstruction of the Wigner function of a pure state can be performed on the observation level which is based on a measurement of a finite number of observables.

We also stress that the choice of a given observation level $\mathcal{O}_{\rm X}$ can be understood as an application of a particular truncation scheme [26] and the difference between the entropy $S_{\rm X}$ and the von Neumann entropy S_0 can be accepted as a measure of the "quality" of the adopted truncation scheme. The smaller this difference more precise the truncation scheme is.

5. Reconstruction of Wigner Functions

5.1. *Coherent States*

The Wigner function $W_{|\alpha\rangle}(\xi)$ of a coherent state $|\alpha\rangle$ on the complete observation level is given by Eq. (2.30) [see Fig. 1a]. Coherent states are uniquely characterized by their amplitude and phase and therefore phase-sensitive observation levels have to be considered for a proper reconstruction of their Wigner functions. In Section 4.1 we have shown that the Wigner function of coherent states can be *completely* reconstructed on the observation level $\mathcal{O}_1$ (see Fig. 1a). Nevertheless it is interesting to understand how Wigner functions of coherent states can be reconstructed on phase-insensitive observation levels.

Observation level $\mathcal{O}_{\rm A}$. The coherent state $|\alpha\rangle$ has a Poissonian photon number distribution and therefore we obtain for the generalized density operator of the coherent state on $\mathcal{O}_{\rm A}$ the expression

$$\hat{\sigma}_{\rm A} = \sum_{n=0}^{\infty} P_n\, |n\rangle\langle n|; \qquad P_n = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!}. \tag{5.1a}$$

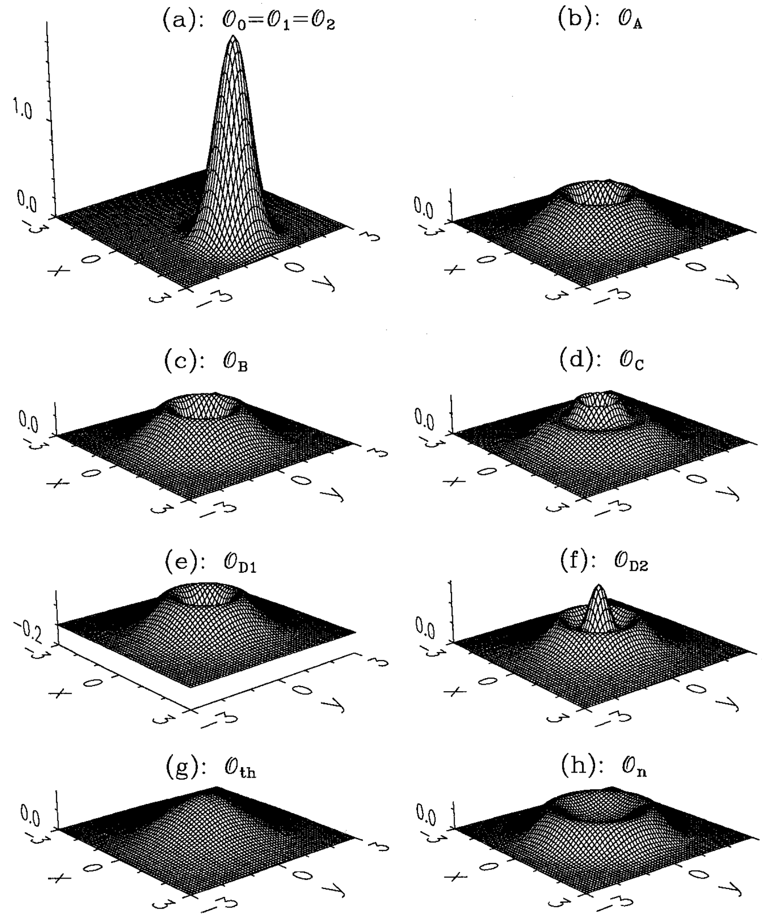

FIG. 1. The reconstructed Wigner functions of the coherent state $|\alpha\rangle$ with $\bar{n}=2$. We consider the observation levels as indicated in the figure.

This density operator describes a phase-diffused coherent state. Eq. (5.1a) can be rewritten in the coherent-state basis

$$\hat{\sigma}_\mathrm{A} = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi\ |\alpha\rangle\langle\alpha|; \qquad \alpha = |\alpha|\, e^{i\phi}. \tag{5.1b}$$

From Eqs. (5.1) it follows that on the observation level $\mathcal{O}_\mathrm{A}$ phase information is completely lost and the corresponding Wigner function can be written as

$$W^{(\mathrm{A})}_{|\alpha\rangle}(\xi) = 2\exp(-2\,|\xi|^2 - |\alpha|^2) \sum_{n=0}^{\infty} \frac{(-|\alpha|^2)^n}{n!}\, \mathscr{L}_n(4\,|\xi|^2), \tag{5.2a}$$

or after some algebra we can find

$$W^{(A)}_{|\alpha\rangle}(\xi) = 2\exp(-2|\xi|^2 - 2|\alpha|^2)\, J_0(4i|\alpha||\xi|), \tag{5.2b}$$

where $J_0(4i|\alpha||\xi|)$ is the Bessel function

$$J_0(4i|\alpha||\xi|) = \sum_{n=0}^{\infty} \frac{(4|\alpha|^2|\xi|^2)^n}{(n!)^2}, \tag{5.3}$$

from which we see that the Wigner function (5.2) is positive. We plot $W^{(A)}_{|\alpha\rangle}(\xi)$ in Fig. 1b. We can understand the shape of $W^{(A)}_{|\alpha\rangle}(\xi)$ if we imagine phase-averaging of the Wigner function $W_{|\alpha\rangle}(\xi)$ [see Fig. 1a]. On the other hand we can represent $W^{(A)}_{|\alpha\rangle}(\xi)$ as a sum of weighted Wigner functions of Fock states [see Eq. (5.2a)]. For the considered coherent state $|\alpha\rangle$ with the mean photon number $\bar{n}=2$ we have $P_1 = P_2 = 2P_0 = 2\exp(-2)$, so the Wigner functions of Fock states $|1\rangle$ and $|2\rangle$ dominantly contribute to $W^{(A)}_{|\alpha\rangle}(\xi)$. On the other hand contribution of the Wigner function of the vacuum state is suppressed and therefore $W^{(A)}_{|\alpha\rangle}(\xi)$ has a local minimum around the origin of the phase space while its maximum is at the same distance from the origin of the phase space as for the Wigner function on the complete observation level [see Fig. 1a]. We note that the Wigner function $W^{(A)}_{|\alpha\rangle}(\xi)$ desribing the phase-difused coherent state has been experimentally reconstructed recently by Raymer *et al.* [17].

Observation level $\mathcal{O}_B$. Let us assume that from a measurement the mean photon number $\bar{n}$ and probabilities P_{2n} are know (see Section 4.2). If the values of P_{2n} are given by Poissonian distribution (5.1), i.e., $P_{2n} = \exp(-\bar{n})\,\bar{n}^{2n}/(2n)!$, then using definitions (4.39) we can find the parameters P_{odd} and $\bar{n}_{\text{odd}}$ to be

$$P_{\text{odd}} = e^{-\bar{n}} \sinh \bar{n}; \qquad \bar{n}_{\text{odd}} = \bar{n}(1 - P_{\text{odd}}), \tag{5.4}$$

The reconstructed probabilities P_{2n+1} are given by Eq. (4.42) and in the limit of large $\bar{n}$ (when $P_{\text{odd}} \to 1/2$ and $\bar{n}_{\text{odd}} \to \bar{n}/2$) they read

$$P_{2n+1} \to \frac{(\bar{n}-1)^n}{(\bar{n}+1)^{n+1}}. \tag{5.5}$$

With the help of the relation (4.37) and explicit expressions for P_{2n} and P_{2n+1} we can evaluate expression for the Wigner function of the coherent state on the observation level $\mathcal{O}_B$. We plot $W^{(B)}_{|\alpha\rangle}(\xi)$ of the coherent state with the mean photon number equal to two ($\bar{n}=2$) in Fig. 1c. In this case P_2 is dominant from which it follows that the Fock state $|2\rangle$ gives a significant contribution into $W^{(B)}_{|\alpha\rangle}(\xi)$ [compare with Fig. 1b].

Observation level $\mathcal{O}_C$. The Wigner function $W^{(C)}_{|\alpha\rangle}(\xi)$ of the coherent state on the observation level $\mathcal{O}_C$ can be reconstructed in exactly same way as on the level $\mathcal{O}_B$. In Fig. 1d we present a result of this reconstruction. On the observation level $\mathcal{O}_C$

the contribution of the vacuum state is more significant than in the case $\mathcal{O}_{\mathrm{B}}$ which is due to the thermal-like photon number distribution P_{2n} on the even-number subspace of the Fock space [see Eq. (4.44)].

Observation level $\mathcal{O}_{\mathrm{D1}}$. We can easily reconstruct the Wigner function of the coherent state at the observation level $\mathcal{O}_{\mathrm{D1}}$. Using general expressions from Section 4.2 we find the expression

$$W_{|\alpha\rangle}^{(\mathrm{D1})}(\xi) = \left(P_0 - \frac{1-P_0}{\tilde{n}}\right) W_{|0\rangle}(\xi) + (1-P_0)\,\frac{\tilde{n}+1}{\tilde{n}}\, W_{\mathrm{th}}(\xi), \tag{5.6}$$

for the Wigner function $W_{|\alpha\rangle}^{(\mathrm{D1})}(\xi)$ [we remind ourselves that for coherent state the parameter P_0 is given by the relation $P_0 = \exp(-\bar{n})$]; where $W_{|0\rangle}(\xi)$ is the Wigner function of the vacuum state given by Eq. (2.30) and $W_{\mathrm{th}}(\xi)$ is the Wigner function of a thermal state (4.10) with an effective number of photons equal to $\tilde{n}$, where

$$\tilde{n} = \frac{\bar{n}}{1-P_0} - 1. \tag{5.7}$$

In particular, from Eqs. (5.6) and (5.7) it follows that

$$\lim_{\bar{n}\to 0} W_{|\alpha\rangle}^{(\mathrm{D1})}(\xi) = W_{|0\rangle}(\xi) \tag{5.8}$$

and simultaneously $S_{\mathrm{D1}} = 0$, which means that the vacuum state can be completely reconstructed on the present observation level. Another result which can be derived from Eq. (5.6) is that if $P_0(2\bar{n}+1) < 1$, then the reconstructed Wigner function $W_{|\alpha\rangle}^{(\mathrm{D1})}(\xi)$ of the coherent state $|\alpha\rangle$ can be negative due to the fact that the contribution of the Fock state $|1\rangle$ is more dominant than the contribution of the vacuum state and then the negativity of the Wigner function $W_{|1\rangle}(\xi)$ results into negative values of $W_{|\alpha\rangle}^{(\mathrm{D1})}(\xi)$. This means that even though the Wigner function of the state itself (i.e., the Wigner function at the complete observation level) is positive, the reconstructed Wigner function can be negative. This is a clear indication that the observation level has to be chosen very carefully and that reconstructed Wigner functions can indicate nonclassical behaviour even in those cases when the measured state itself does not exhibit nonclassical effects. In Fig.1e we plot the Wigner function $W_{|\alpha\rangle}^{(\mathrm{D1})}(\xi)$ of the coherent state which illustrates this effect.

Observation level $\mathcal{O}_{\mathrm{D2}}$. If the mean photon number $\bar{n}$ is an integer, then one may consider the observation level $\mathcal{O}_{\mathrm{D2}}$. The Wigner function of the coherent state at this observation level for which $P_{\bar{n}} = \exp(-\bar{n})\,\bar{n}^{\bar{n}}/(\bar{n}!)$ reads

$$W_{|\alpha\rangle}^{(\mathrm{D2})}(\xi) = \left(1 - \frac{1+\bar{n}}{Z_{\mathrm{D2}}}\right) W_{|\bar{n}\rangle}(\xi) + \frac{\bar{n}+1}{Z_{\mathrm{D2}}}\, W_{\mathrm{th}}(\xi), \tag{5.9}$$

where $W_{|\bar{n}\rangle}(\xi)$ is the Wigner function of the Fock state $|\bar{n}\rangle$ and $W_{\mathrm{th}}(\xi)$ is the Wigner function of the thermal state with the mean photon number equal to $\bar{n}$. The partition function Z_{D2} is given by the relation (4.57b). The Wigner function (5.9) is plotted in Fig. 1f. From this figure we see that the vacuum state $|0\rangle$ (due to the thermal-like character of the reconstructed photon number distribution) and the Fock state $|2\rangle$ (as a consequence of the measurement) dominantly contribute to $W_{|\alpha\rangle}^{(D2)}(\xi)$.

5.2. *Squeezed Vacuum*

The Wigner function of the squeezed vacuum state (2.41) on the complete observation level $\mathcal{O}_0$ is given by Eq. (2.44) and is plotted (in the complex ξ phase space) in Fig. 2a. This is a Gaussian function, which carries phase information associated with the phase of squeezing. On the thermal observation level $\mathcal{O}_{th}$ which is characterized only by the mean photon number $\bar{n}$ the reconstructed Wigner function of the squeezed vacuum state is a rotationally symmetrical Gaussian function centered at the origin of the phase space [see Eq. (4.10) and Fig. 1g]. On the observation level $\mathcal{O}_1$ the reconstructed Wigner function is the same as on the thermal observation

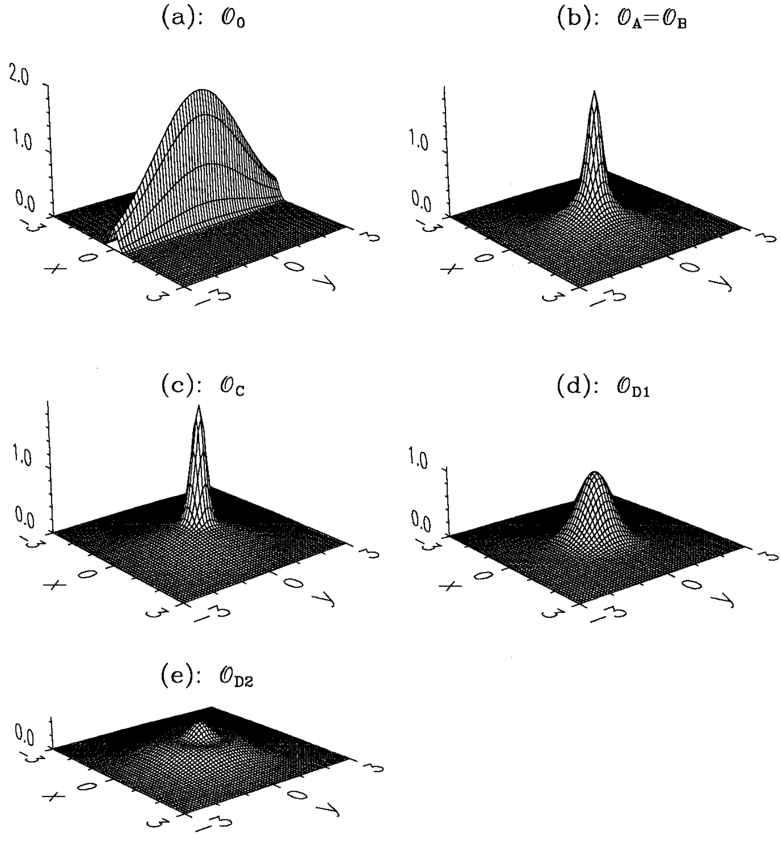

FIG. 2. The reconstructed Wigner functions of the squeezed vacuum state $|\eta\rangle$ with $\bar{n}=2$. We consider the observation levels as indicated in the figure.

level because the mean amplitudes $\langle \hat{a} \rangle$ and $\langle \hat{a}^\dagger \rangle$ are equal to zero. On the other hand, the Wigner function of the squeezed vacuum can be completely reconstructed on the observation level $\mathcal{O}_2$. To see this we evaluate the entropy S_2 for the squeezed vacuum state [see Eq. (4.24)]. The parameters M and N can be expressed in terms of the squeezing parameter η (we assume η to be real) as

$$N = \frac{\eta^2}{1-\eta^2}; \qquad M = \frac{\eta}{1-\eta^2}, \tag{5.10}$$

so that $N(N+1) = M^2$. Consequently the parameter χ given by Eq. (4.21b) is equal to zero from which it follows that S_2 for the squeezed vacuum is equal to zero.

Observation level $\mathcal{O}_{\mathrm{A}}$. The squeezed vacuum state (2.41a) is characterized by the oscillatory photon number distribution P_n:

$$P_{2n} = (1-\eta^2)^{1/2} \frac{(2n)!}{[2^n n!]^2} \eta^{2n}; \qquad P_{2n+1} = 0. \tag{5.11}$$

Using Eq. (4.37) we can express the Wigner function $W^{(A)}_{|\eta\rangle}(\xi)$ of the squeezed vacuum on the observation level $\mathcal{O}_{\mathrm{A}}$ as

$$W^{(A)}_{|\eta\rangle}(\xi) = 2(1-\eta^2)^{1/2} e^{-2|\xi|^2} \sum_{n=0}^{\infty} \frac{(2n)!\, \eta^{2n}}{2^{2n}(n!)^2} \mathcal{L}_{2n}(4|\xi|^2). \tag{5.12}$$

Taking into account that the Wigner function on the observation level $\mathcal{O}_{\mathrm{A}}$ can be obtained as the phase-averaged Wigner function on the complete observation level, we can rewrite (5.12) as

$$W^{(A)}_{|\eta\rangle}(\xi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} W_{|\eta\rangle}(\xi)\, d\phi; \qquad \xi = |\xi|\, e^{i\phi}. \tag{5.13}$$

If we insert the explicit expression for $W_{|\eta\rangle}(\xi)$ [see Eq. (2.44)] into Eq. (5.13) we obtain

$$W^{(A)}_{|\eta\rangle}(\xi) = 2 \exp\left[-\left(\frac{|\xi|^2}{2\sigma_q^2} + \frac{|\xi|^2}{2\sigma_p^2} \right) \right] I_0 \left(\frac{|\xi|^2}{2\sigma_q^2} - \frac{|\xi|^2}{2\sigma_p^2} \right), \tag{5.14}$$

where $I_0(x)$ is the modified Bessel function. We plot this Wigner function in Fig. 2b. We see that $W^{(A)}_{|\eta\rangle}(\xi)$ is not negative and that it is much narrower in the vicinity of the origin of the phase space than the Wigner function of the vacuum state (compare with Fig. 1a). Nevertheless the total width of Wigner function $W^{(\mathrm{A})}_{|\eta\rangle}(\xi)$ is much larger than the width of the Wigner function of the vacuum state.

Observation level $\mathcal{O}_{\mathrm{B}}$. Due to the fact that for the squeezed vacuum state we have $\sum_n P_{2n} = 1$, the Wigner function of the squeezed vacuum state on the observation level $\mathcal{O}_{\mathrm{B}}$ is equal to the Wigner function on the observation level $\mathcal{O}_{\mathrm{A}}$, i.e., $W^{(B)}_{|\eta\rangle}(\xi) = W^{(A)}_{|\eta\rangle}(\xi)$.

Observation level $\mathcal{O}_{\mathrm{C}}$. For the squeezed vacuum state all meanvalues P_{2n+1} are equal to zero and therefore $\sum_n P_{2n+1}=0$. From this fact and from the knowledge of the mean photon number $\bar{n}$ we can reconstruct the Wigner function $W^{(C)}_{|\eta\rangle}(\xi)$ in the form [see Section 4.2.3]

$$W^{(C)}_{|\eta\rangle}(\xi)=\frac{4\mathrm{e}^{-2|\xi|^2}}{\bar{n}+2}\sum_{k=0}^{\infty}\left(\frac{\bar{n}}{\bar{n}+2}\right)^k \mathscr{L}_{2k}(4|\xi|^2), \tag{5.15}$$

where $\bar{n}$ is the mean photon number in the squeezed vacuum state. We plot the Wigner function $W^{(C)}_{|\eta\rangle}(\xi)$ in Fig. 2c. This Wigner function is very similar to the Wigner function on the observation level $\mathcal{O}_{\mathrm{A}}$ [see Fig. 2b] which reflects the fact that the photon number distribution of the squeezed vacuum state has a thermal-like character on the even-number subspace of the Fock space.

Observation level $\mathcal{O}_{\mathrm{D1}}$. With the help of the general formalism presented in Section 4.2.4 we can express the Wigner function $W^{(D1)}_{|\eta\rangle}(\xi)$ of the squeezed vacuum state on the observation level $\mathcal{O}_{\mathrm{D1}}$ in the form given by Eq. (5.6) with

$$P_0=(1-\eta^2)^{1/2}=(\bar{n}+1)^{-1/2} \qquad \text{and} \qquad \tilde{n}=\frac{\bar{n}}{1-(1+\bar{n})^{-1/2}}-1. \tag{5.16}$$

We plot the Wigner function $W^{(D1)}_{|\eta\rangle}(\xi)$ in Fig. 2d from which the dominant contribution of the vacuum state is transparent which is due to the fact that the squeezed vacuum state has a thermal-like photon number distribution (for more details see Sec. 6).

Observation level $\mathcal{O}_{\mathrm{D2}}$. If we consider $\bar{n}$ to be an *even* integer, then the Wigner function $W^{(D2)}_{|\eta\rangle}(\xi)$ of the squeezed vacuum state on $\mathcal{O}_{\mathrm{D2}}$ is given by Eq. (5.9). The partition function Z_{D2} is given by Eq. (4.57b) where

$$P_{\bar{n}}=\frac{\bar{n}!}{2^{\bar{n}}[(\bar{n}/2)!]^2}\frac{\bar{n}^{\bar{n}/2}}{(1+\bar{n})^{(1+\bar{n})/2}}. \tag{5.17}$$

We plot this Wigner function in Fig. 2e. It has a thermal-like character [compare with Fig. 1g] but contribution of the Fock state $|\bar{n}=2\rangle$ is more dominant compared with the proper thermal distribution. If $\bar{n}$ is an *odd* integer, then $P_{\bar{n}}=0$ and the corresponding Wigner function can be again reconstructed with the help of Eqs. (5.9) and (4.57b).

5.3. *Even Coherent State*

We plot the Wigner function of the even coherent state on the complete observation level in Fig. 3a. Two contributions of coherent component state $|\alpha\rangle$ and $|-\alpha\rangle$ as well as the interference peak around the origin of the phase space are transparent in this figure. As in the case of the squeezed vacuum state, the mean amplitude $\langle\hat{a}\rangle$

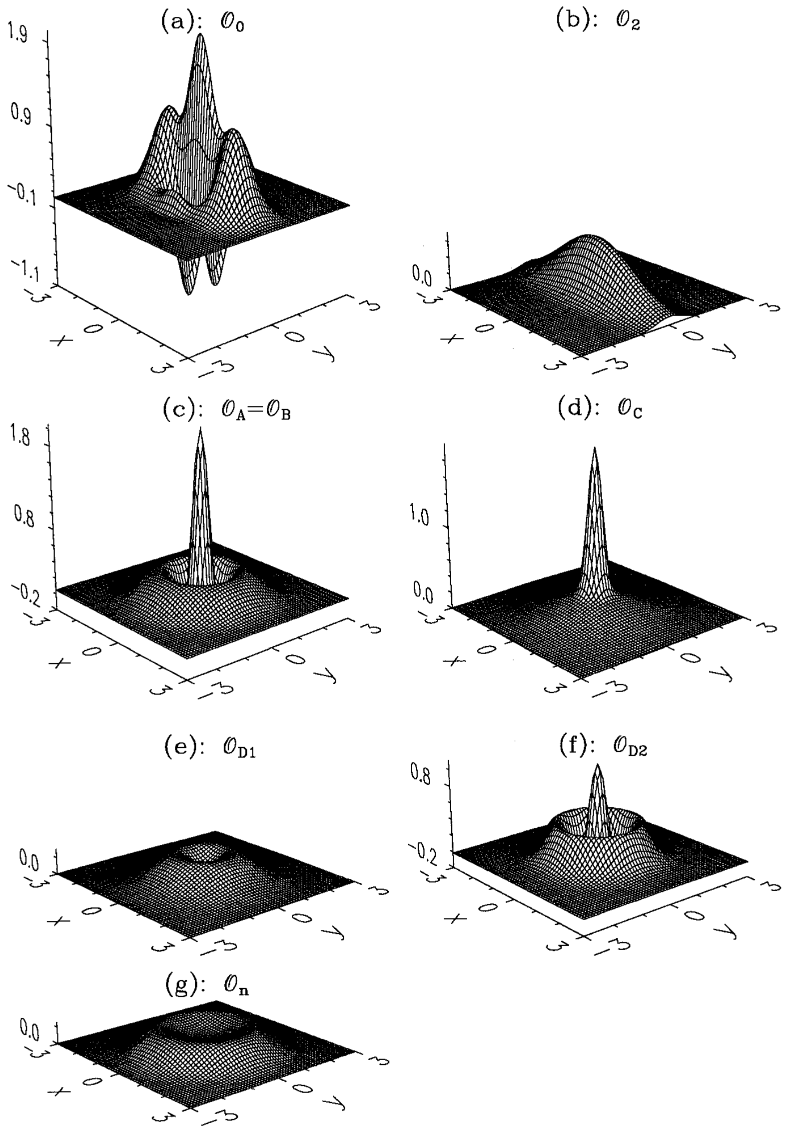

FIG. 3. The reconstruced Wigner functions of the even coherent state $|\alpha_e\rangle$ with $\bar{n}=2$. We consider the observation levels as indicated in the figure.

of the even coherent state is equal to zero and therefore the Wigner function $W^{(1)}_{|\alpha_e\rangle}(\xi)$ of the even coherent state on the observation level $\mathcal{O}_1$ is equal to the thermal Wigner function given by Eq. (4.10).

Observation level $\mathcal{O}_2$. Using general expressions from Section 4.1.2 we can express the Wigner function $W^{(2)}_{|\alpha_e\rangle}(\xi)$ of the even coherent state on the observation level $\mathcal{O}_2$ as

$$W^{(2)}_{|\alpha_e\rangle}(\xi)=\frac{1}{[(N+1/2)^2-M^2]^{1/2}}\exp\left[-\frac{\xi_x^2}{[(N+1/2)+M]}-\frac{\xi_y^2}{[(N+1/2)-M]}\right], \tag{5.18a}$$

where $\xi=\xi_x+i\xi_y$, and the parameters N and M read

$$N=\alpha^2\tanh\alpha^2;\qquad M=\alpha^2. \tag{5.18b}$$

We plot the Wigner function $W^{(2)}_{|\alpha_e\rangle}$ in Fig. 3b. This Wigner function is slightly "squeezed" in the ξ_y-direction and stretched in the ξ_x-direction. Nevertheless, the reconstructed Wigner function is different from the Wigner function of the squeezed vacuum state [compare with Fig. 2a].

Observation level $\mathcal{O}_{\mathrm{A}}$. The photon number distribution of the even coherent state is given by the relation (we assume α to be real)

$$P_{2n}=\frac{1}{\cosh\alpha^2}\frac{\alpha^{4n}}{(2n)!};\qquad P_{2n+1}=0, \tag{5.19}$$

so the corresponding Wigner function can be expressed as Eq. (4.37). We can also express $W^{(A)}_{|\alpha_e\rangle}(\xi)$ as the phase averaged Wigner function of the even coherent state $W_{|\alpha_e\rangle}(\xi)$ given by Eq. (2.47a). After some algebra we find that $W^{(A)}_{|\alpha_e\rangle}(\xi)$ can be written in a closed form

$$W^{(A)}_{|\alpha_e\rangle}(\xi)=\frac{e^{-2|\xi|^2}}{\cosh\alpha^2}\left[e^{-\alpha^2}J_0(4i\alpha|\xi|)+e^{\alpha^2}J_0(4\alpha|\xi|)\right]. \tag{5.20}$$

We plot the Wigner function $W^{(A)}_{|\alpha_e\rangle}(\xi)$ in Fig. 3c. From this figure the dominant contribution of the Fock state $|2\rangle$ is transparent (in the present case we have $P_0\simeq 2\exp(-2)$, $P_2=2P_0$, and $P_4=2P_0/3$, while all other probabilities P_n are much smaller) which results in negative Wigner function.

Observation level $\mathcal{O}_{\mathrm{B}}$. Due to the fact that the even coherent state is expressed as a superposition of only even Fock states, i.e., $\sum_n P_{2n}=1$, the Wigner functions on the observation levels $\mathcal{O}_{\mathrm{A}}$ and $\mathcal{O}_{\mathrm{B}}$ are equal, i.e., $W^{(B)}_{|\alpha_e\rangle}(\xi)=W^{(A)}_{|\alpha_e\rangle}(\xi)$.

Observation level $\mathcal{O}_{\mathrm{C}}$. As a consequence of the fact that for the even coherent state all meanvalues P_{2n+1} are equal to zero the information available for the reconstruction of the Wigner function $W^{(C)}_{|\alpha_e\rangle}(\xi)$ is the same as in the case of the reconstruction of the Wigner function of the squeezed vacuum state on the observation level $\mathcal{O}_{\mathrm{C}}$. Therefore, the Wigner function $W^{(C)}_{|\alpha_e\rangle}(\xi)$ has exactly the same form as for the squeezed vacuum state with the same mean photon number $\bar{n}$ [see Fig. 3d and Fig. 2c].

Observation level $\mathcal{O}_{\mathrm{D1}}$. The Wigner function $W^{(D1)}_{|\alpha_e\rangle}(\xi)$ of the even coherent state on the observation level $\mathcal{O}_{\mathrm{D1}}$ is given by Eq. (5.6) with

$$P_0 = \frac{1}{\cosh \alpha^2}; \qquad \tilde{n} = \frac{\alpha^2 \sinh \alpha^2}{\cosh \alpha^2 - 1} - 1. \tag{5.21}$$

We plot the Wigner function $W^{(D1)}_{|\alpha_e\rangle}(\xi)$ in Fig. 3e. This Wigner function has a thermal-like character except the fact that the contribution of the vacuum state is slightly suppressed.

Observation level $\mathcal{O}_{\mathrm{D2}}$. Analogously we can find the Wigner function $W^{(D2)}_{|\alpha_e\rangle}(\xi)$. If we consider $\bar{n}$ to be an *even* integer, then the Wigner function $W^{(D2)}_{|\eta\rangle}(\xi)$ of the even coherent state on $\mathcal{O}_{\mathrm{D2}}$ is given by Eq. (5.9) and Eq. (4.57b) where

$$P_{\bar{n}} = \frac{1}{\cosh \alpha^2} \frac{\alpha^{2\bar{n}}}{\bar{n}!}, \tag{5.22}$$

and if $\bar{n}$ is an odd integer then $P_{\bar{n}} = 0$. We plot $W^{(D2)}_{|\alpha_e\rangle}(\xi)$ in Fig. 3f. From our previous discussion it is clear that in the present case the vacuum state and the Fock state $|2\rangle$ dominantly contribute to $W^{(D2)}_{|\alpha_e\rangle}(\xi)$ (similarly as on the observation level $\mathcal{O}_{\mathrm{A}}$ - see Fig. 3c).

5.4. *Odd Coherent State*

We present the Wigner function of the odd coherent state with the mean photon number equal to two in Fig. 4a. The mean amplitude $\langle \hat{a} \rangle$ of the odd coherent state is equal to zero and therefore the Wigner function $W^{(1)}_{|\alpha_o\rangle}(\xi)$ of this state on the observation level $\mathcal{O}_1$ is equal to the thermal Wigner function given by Eq. (4.10) [see Fig. 1g].

Observation level $\mathcal{O}_2$. Using general expressions from Section 4.1.2 we find that the Wigner function $W^{(2)}_{|\alpha_o\rangle}(\xi)$ of the odd coherent state on the observation level $\mathcal{O}_2$ is the same as for the even coherent state [see Eq. (5.18a)] but the parameters N and M in the present case read

$$N = \alpha^2 \coth \alpha^2; \qquad M = \alpha^2. \tag{5.23}$$

We plot the Wigner function $W^{(2)}_{|\alpha_o\rangle}(\xi)$ in Fig. 4b. This is a "squeezed"-Gaussian function similar to the Wigner function of the even coherent state on the same observation level [see Fig. 3b and discussion in the previous section].

Observation level $\mathcal{O}_{\mathrm{A}}$. The photon number distribution of the odd coherent state is given by the relation (we assume α to be real)

$$P_{2n} = 0; \qquad P_{2n+1} = \frac{1}{\sinh \alpha^2} \frac{(\alpha^2)^{2n+1}}{(2n+1)!}. \tag{5.24}$$

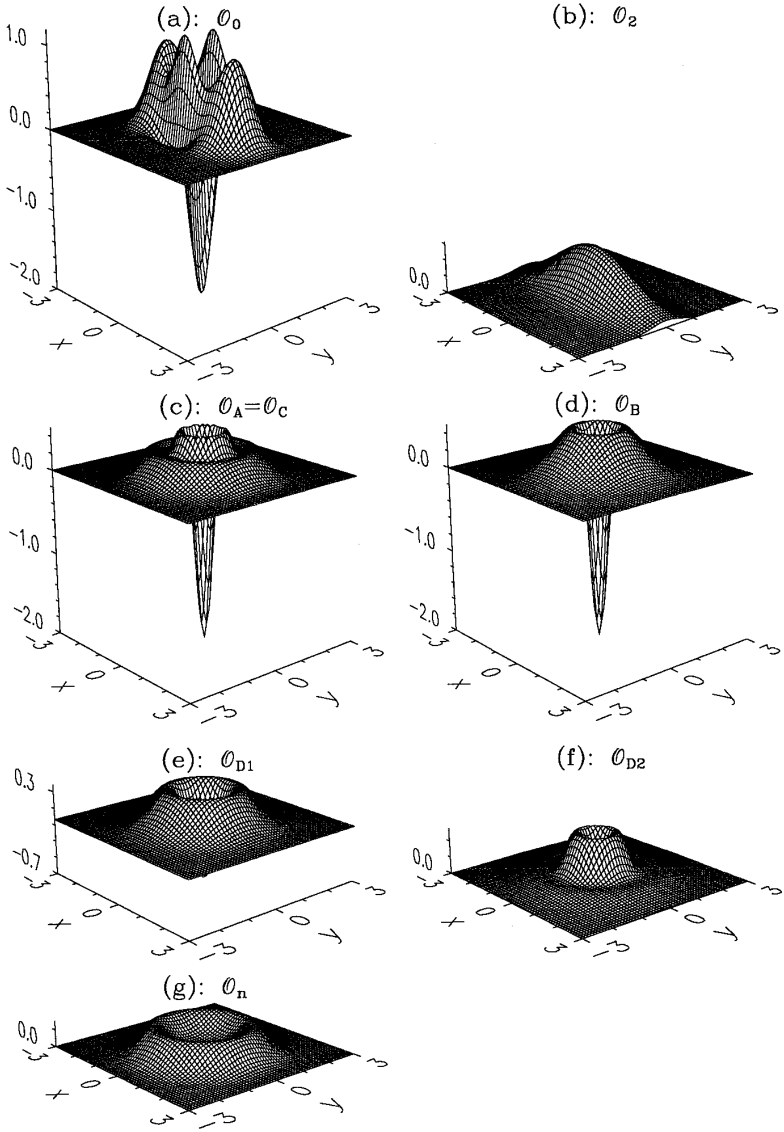

FIG. 4. The reconstructed Wigner functions of the odd coherent state $|\alpha_o\rangle$ with $\bar{n}=2$. We consider the observation levels as indicated in the figure.

Consequently, the Wigner function $W^{(A)}_{|\alpha_o\rangle}(\xi)$ can be expressed as (4.37). Alternatively, if we use the fact that $W^{(A)}_{|\alpha_o\rangle}(\xi)$ is equal to the phase averaged Wigner function of the odd coherent state $W_{|\alpha_o\rangle}(\xi)$ given by Eq. (2.47b), then we can write

$$W^{(A)}_{|\alpha_o\rangle}(\xi)=\frac{e^{-2|\xi|^2}}{\sinh\alpha^2}\left[e^{-\alpha^2}J_0(4i\alpha|\xi|)-e^{\alpha^2}J_0(4\alpha|\xi|)\right]. \tag{5.25}$$

This function is always negative in the origin of the phase space. We plot the Wigner function $W^{(A)}_{|\alpha_o\rangle}(\xi)$ in Fig. 4c. In the present case $P_0 = P_2 = 0$ and the P_1 is the largest probability therefore the contribution of the Fock state $|1\rangle$ in $W^{(A)}_{|\alpha_o\rangle}(\xi)$ is the most dominant which is clearly seen from Fig. 4c. We also note that, in general, any superposition of odd Fock states has a negative Wigner function on the observation level $\mathcal{O}_{\mathrm{A}}$.

Observation level $\mathcal{O}_{\mathrm{B}}$. For the odd coherent state all meanvalues P_{2n} are equal to zero. Taking into account this information and the information about the mean photon number we reconstruct the Wigner function $W^{(B)}_{|\alpha_o\rangle}(\xi)$ in the form (for details see Section 4.4.2)

$$W^{(B)}_{|\alpha_o\rangle}(\xi) = -\frac{4e^{-2|\xi|^2}}{\bar{n}+1} \sum_{k=0}^{\infty} \left(\frac{\bar{n}-1}{\bar{n}+1}\right)^k \mathcal{L}_{2k+1}(4|\xi|^2), \tag{5.26}$$

where $\bar{n} = \alpha^2 \coth \alpha^2$. We plot this Wigner function in Fig. 4d. In the present case the dominant contribution of the Fock state $|1\rangle$ is seen ($P_0 = P_2 = 0$ and due to the thermal-like photon number distribution on the odd-number subspace of the Fock space P_3 is much smaller than P_1). We can conclude, that any superposition of odd Fock states on the observation level $\mathcal{O}_{\mathrm{B}}$ has the Wigner function given by Eq. (5.26), i.e., superpositions of odd Fock states are indistinguishable on $\mathcal{O}_{\mathrm{B}}$.

Observation level $\mathcal{O}_{\mathrm{C}}$. Due to the fact that the odd coherent state is expressed as a superposition of only odd Fock states, i.e., $\sum_n P_{2n+1} = 1$, the Wigner functions on the observation levels $\mathcal{O}_{\mathrm{C}}$ and $\mathcal{O}_{\mathrm{A}}$ are equal, i.e., $W^{(C)}_{|\alpha_o\rangle}(\xi) = W^{(A)}_{|\alpha_o\rangle}(\xi)$.

Observation level $\mathcal{O}_{\mathrm{D1}}$. The Wigner function $W^{(D1)}_{|\alpha_o\rangle}(\xi)$ of the odd coherent state on the observation level $\mathcal{O}_{\mathrm{D1}}$ is given by the following relation [we remind us that for the odd coherent state we have $P_0 = 0$]

$$W^{(D1)}_{|\alpha_o\rangle}(\xi) = -\frac{1}{\bar{n}-1} W_{|0\rangle}(\xi) + \frac{\bar{n}}{\bar{n}-1} W_{\mathrm{th}}(\xi), \tag{5.27}$$

where $\bar{n}$ is the mean photon number in the odd coherent state; $W_{|0\rangle}(\xi)$ is the Wigner function of the vacuum state and $W_{\mathrm{th}}(\xi)$ is the thermal Wigner function for the state with $\bar{n}-1$ photons. We note that from Eq. (5.27) it follows that

$$\lim_{\bar{n}\to 1} W^{(D1)}_{|\alpha_o\rangle}(\xi) = W_{|1\rangle}(\xi), \tag{5.28}$$

We plot the Wigner function $W^{(D1)}_{|\alpha_o\rangle}(\xi)$ in Fig. 4e. Compared with Fig. 4d we see that the contribution of the Fock state $|1\rangle$ on the observation level $\mathcal{O}_{\mathrm{D1}}$ is smaller than on $\mathcal{O}_{\mathrm{B}}$. This is due to the fact that on the present observation level P_2 is not equal to zero.

Observation level $\mathcal{O}_{\mathrm{D2}}$. Reconstruction of the Wigner function $W^{(D2)}_{|\alpha_o\rangle}(\xi)$ is straightforward. For the odd coherent state it is valid that if $\bar{n}$ is an odd integer, then

$$P_{\bar{n}} = \frac{1}{\sinh \alpha^2} \frac{\alpha^{2\bar{n}}}{\bar{n}!}, \tag{5.29}$$

and the Wigner function is given by Eq. (5.9). On the other hand if $\bar{n}$ is an *even* integer, then $P_{\bar{n}} = 0$ and we again use Eq. (5.9) for the reconstruction of the Wigner function $W^{(D2)}_{|\eta\rangle}(\xi)$. We plot this Wigner function in Fig. 4f. Even though on this observation level $P_2 = 0$ the contribution from the vacuum state is significant and therefore $W^{(D1)}_{|\alpha_o\rangle}(\xi)$ is not negative in the present case.

5.5. *Fock State*

Mean values of the operators $\hat{a}^k$ in the Fock state are equal to zero, therefore the Wigner functions $W^{(1)}_{|n\rangle}(\xi)$ and $W^{(2)}_{|n\rangle}(\xi)$ of the Fock state $|n\rangle$ on the observation levels $\mathcal{O}_1$ and $\mathcal{O}_2$, respectively, are equal to the thermal Wigner function given by Eq. (4.10) [see Fig. 5b]. On the other hand the Shannon entropy of the Fock state is equal to zero, therefore this state can be completely reconstructed on the observation level $\mathcal{O}_{\mathrm{A}}$ [see Fig. 5a for the Wigner function of the Fock state $|2\rangle$].

Observation level $\mathcal{O}_{\mathrm{B}}$. If the Fock state has an even number of photons then it can also be completely reconstructed on the observation level $\mathcal{O}_{\mathrm{B}}$. But if the number

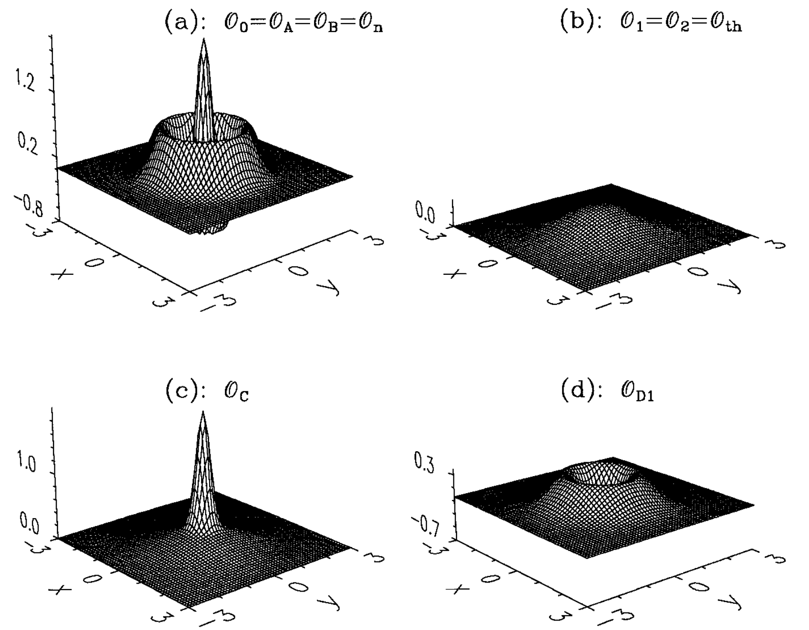

FIG. 5. The reconstructed Wigner functions of the Fock state $|n=2\rangle$. We consider the observation levels as indicated in the figure.

of photons of the Fock state is odd then the Wigner function of this Fock state on $\mathcal{O}_{\mathrm{B}}$ is given by the relation (5.26) with $\bar{n}=n$.

Observation level $\mathcal{O}_{\mathrm{C}}$. If the number of photons of the Fock state is odd then the corresponding Wigner function can be completely reconstructed on the observation level $\mathcal{O}_{\mathrm{C}}$. If the number of photons is even, then the Wigner function $W^{(C)}_{|n\rangle}(\xi)$ is given by Eq. (5.15) with $\bar{n}=n$. We plot $W^{(C)}_{|n\rangle}(\xi)$ in Fig. 5c. This Wigner function is the same as for the squeezed vacuum state $W^{(C)}_{|\eta\rangle}(\xi)$ and the even coherent state $W^{(C)}_{|\alpha_e\rangle}(\xi)$ with the same mean photon number [see Figs. 2c and 3c]. More generally, all superpositions of even Fock states with the same mean photon number are indistinguishable on $\mathcal{O}_{\mathrm{C}}$.

Observation level $\mathcal{O}_{\mathrm{D1}}$. If the Fock state under consideration is the vacuum state then it can be completely reconstructed on the observation level $\mathcal{O}_{\mathrm{D1}}$. If the number of photons is larger than zero, then $P_0=0$ and the corresponding Wigner function is given by Eq. (5.27) with $\bar{n}=n$. We plot $W^{(D1)}_{|n=2\rangle}(\xi)$ in Fig. 5d.

Observation level $\mathcal{O}_{\mathrm{D2}}$. On this observation level the Wigner function of the Fock state $|n\rangle$ can be always completely reconstructed, because this observation level is defined in such way that $P_n=1$.

6. Reconstruction of Eigenstates of Observables

From the von Neumann theory of measurement [13] it follows that the necessary and sufficient condition that $|A\rangle$ is an eigenstate of the observable $\hat{A}$ is

$$\langle A|\,(\hat{A}-\langle\hat{A}\rangle)^2\,|A\rangle=0. \tag{6.1}$$

From this condition it follows that there has to exist an observation level on which we can reconstruct the Wigner function of the state $|A\rangle$ via the measurement of the observables $\hat{A}$ and $\hat{A}^2$.

6.1. *Observation level* $\mathcal{O}_{\mathrm{q}}\equiv\{\hat{q},\hat{q}^2\}$.

To be more specific, let us consider a reconstruction of the eigenstate $|\bar{q}\rangle$ [see Appendix B] of the position operator $\hat{q}$. To do so, we will utilize the observation level $\mathcal{O}_{\mathrm{q}}\equiv\{\hat{q},\hat{q}^2\}$. The generalized canonical density operator in this case reads

$$\hat{\sigma}_{\mathrm{q}}=\frac{1}{Z_{\mathrm{q}}}\exp[-\lambda_1\hat{q}-\lambda_2\hat{q}^2]. \tag{6.2}$$

We note that due to the fact that the position operator $\hat{q}$ can be expressed in terms of the photon creation and annihilation operators [i.e., $\hat{q}=(\hat{a}^\dagger+\hat{a})\sqrt{\hbar/2}$] the observation level $\mathcal{O}_{\mathrm{q}}$ is closely related to the observation level $\mathcal{O}_2$. More precisely, $\mathcal{O}_{\mathrm{q}}$ represents a reduction of $\mathcal{O}_2$.

We can rewrite the density operator $\hat{\sigma}_{\mathrm{q}}$ given by Eq. (6.2) as

$$\hat{\sigma}_{\mathrm{q}} = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dq\, P(q)\, |q\rangle\langle q|, \tag{6.3}$$

where a distribution $P(q) = \sqrt{2\pi\hbar}\, \langle q|\, \hat{\sigma}_{\mathrm{q}}\, |q\rangle$ [see Eq. (2.21a)] has a Gaussian form

$$P(q) = \frac{1}{Z_{\mathrm{q}}} \exp[-\lambda_1 q - \lambda_2 q^2], \tag{6.4a}$$

and is normalized to unity

$$\frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dq\, P(q) = 1. \tag{6.4b}$$

The corresponding partition function Z_{q} can be evaluated explicitly in a form

$$Z_{\mathrm{q}} = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dq \exp[-\lambda_1 q - \lambda_2 q^2] = \frac{1}{\sqrt{2\hbar\lambda_2}} \exp\left(\frac{\lambda_1^2}{4\lambda_2}\right). \tag{6.5}$$

Lagrange multipliers λ_1 and λ_2 are defined by the relations (see Section 2)

$$\langle \hat{q} \rangle = -\frac{\partial \ln Z_{\mathrm{q}}}{\partial \lambda_1} = -\frac{\lambda_1}{2\lambda_2}; \tag{6.6a}$$

$$\langle \hat{q}^2 \rangle = -\frac{\partial \ln Z_{\mathrm{q}}}{\partial \lambda_2} = \frac{1}{2\lambda_2} + \frac{\lambda_1^2}{4\lambda_2^2}, \tag{6.6b}$$

from which we find

$$\lambda_1 = -\frac{\bar{q}}{\hbar\sigma_q^2}; \qquad \lambda_2 = \frac{1}{2\hbar\sigma_q^2}, \tag{6.7}$$

where we have used the notation

$$\bar{q} \equiv \langle \hat{q} \rangle; \qquad \langle \hat{q}^2 \rangle - \langle \hat{q} \rangle^2 = \hbar\sigma_q^2. \tag{6.8}$$

Consequently we can write an explicit expression for the partition function (6.5)

$$Z_{\mathrm{q}} = \sigma_q \exp\left[\frac{(\bar{q})^2}{2\hbar\sigma_q^2}\right], \tag{6.9}$$

and then the probability distribution $P(q)$ given by Eq. (6.4) reads [see also Appendix B]

$$P(q) = \frac{1}{\sigma_q} \exp\left[-\frac{(q-\bar{q})^2}{2\hbar\sigma_q^2}\right]. \tag{6.10}$$

The generalized density operator $\hat{\sigma}_{\mathrm{q}}$ [see Eq. (6.2)] now takes the form

$$\hat{\sigma}_{\mathrm{q}} = \frac{1}{\sigma_q} \exp\left[-\frac{(\hat{q}-\bar{q})^2}{2\hbar\sigma_q^2} \right], \tag{6.11}$$

and the corresponding entropy reads

$$S_{\mathrm{q}} = -k_{\mathrm{B}} \operatorname{Tr}[\hat{\sigma}_{\mathrm{q}} \ln \hat{\sigma}_{\mathrm{q}}] = -\frac{k_{\mathrm{B}}}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dq\, P(q) \ln P(q) = \frac{k_{\mathrm{B}}}{2} + k_B \ln \sigma_q. \tag{6.12}$$

The generalized canonical density operator $\hat{\sigma}_{\mathrm{q}}$ given by Eq. (6.11) does not provide us with a sufficient information to reconstruct the Wigner function $W(q, p)$ in the (q, p)-phase space. Actually, this is not surprising, because the observation level $\mathcal{O}_{\mathrm{q}}$ is defined in such way that information just about *one* canonical observable (i.e., $\hat{q}$) is available while no information about the conjugated observable $\hat{p}$ is at disposal. Nevertheless, in the limit $\sigma_q \to 0$, i.e., when $\langle \hat{q}^2 \rangle - \langle \hat{q} \rangle^2 \to 0$ and $P(q) \to \sqrt{2\pi\hbar}\, \delta(q-\bar{q})$ [see Eq. (6.10)] we find from Eq. (6.11) that

$$\lim_{\sigma_q \to 0} \hat{\sigma}_{\mathrm{q}} = |\bar{q}\rangle\langle\bar{q}|, \tag{6.13}$$

which means that on the observation level $\mathcal{O}_{\mathrm{q}}$ we can *completely* reconstruct the position state $|\bar{q}\rangle$. In this case the Wigner function is given by Eq. (B.3) in the limit of infinite squeezing. We should also point out that in the limit $\sigma_q \to 0$ the entropy S_{q} is equal to $-\infty$. This is related to the fact that in the limit $\sigma_q \to 0$ the distribution $P(q)$ given by Eq. (6.10) has a form of the δ-function analogous to a probability density distribution of a classical continuous variable for which entropy can really take a value equal to $-\infty$ [39].

6.2. *Observation Level* $\mathcal{O}_{\mathrm{n}} \equiv \{\hat{n}, \hat{n}^2\}$

From our previous discussion it follows that one cannot reconstruct the Wigner function $W(q, p)$ in the (q, p)-phase space providing information just about one of the two observable $\hat{q}$ and $\hat{p}$ is available. On the other hand, the observation level $\mathcal{O}_{\mathrm{q}}$ is suitable for a complete reconstruction of the eigenstate of the position operator.

Now we will assume the phase-insensitive observation level $\mathcal{O}_{\mathrm{n}}$ which is related to a measurement of the observables $\hat{n}$ and $\hat{n}^2$. Due to the fact that the operators $\hat{n}$ and $\hat{n}^2$ can be expressed in terms of powers of the position and momentum operators we expect that the Wigner function $W^{(\mathrm{n})}(q, p)$ in the (q, p)-phase space on observation level $\mathcal{O}_{\mathrm{n}}$ can be reconstructed.

The generalized canonical density operator $\hat{\sigma}$ on the observation level $\mathcal{O}_{\mathrm{n}}$ reads

$$\hat{\sigma}_{\mathrm{n}} = \frac{1}{Z_{\mathrm{n}}} \exp[-\lambda_1 \hat{n} - \lambda_2 \hat{n}^2] = \frac{1}{Z_{\mathrm{n}}} \sum_{n=0}^{\infty} \exp[-\lambda_1 n - \lambda_2 n^2]\, |n\rangle\langle n|. \tag{6.14}$$

The Lagrange multipliers are determined by the relations

$$\langle \hat{n} \rangle = -\frac{\partial \ln Z_{\mathrm{n}}}{\partial \lambda_1} = \sum_{m=0}^{\infty} m P_m; \tag{6.15a}$$

$$\langle \hat{n}^2 \rangle = -\frac{\partial \ln Z_{\mathrm{n}}}{\partial \lambda_2} = \sum_{m=0}^{\infty} m^2 P_m, \tag{6.15b}$$

where

$$P_m = \frac{1}{Z_{\mathrm{n}}} \exp[-\lambda_1 m - \lambda_2 m^2]; \tag{6.16a}$$

and

$$Z_{\mathrm{n}} = \sum_{m=0}^{\infty} \exp[-\lambda_1 m - \lambda_2 m^2]. \tag{6.16b}$$

From Eqs. (6.15) it follows that if $\langle \hat{n} \rangle = N$ is an integer, then in the limit $\sigma_n \to 0_+$ (where $\sigma_n^2 \equiv \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2$) $\lambda_1 = -2N\lambda_2$ and λ_2 tends to infinity. Simultaneously

$$P_m \to \delta_{m,N}, \tag{6.17}$$

which means that in this case $\hat{\sigma}_{\mathrm{n}} \to |N\rangle\langle N|$. In other words, on the observation level $\mathcal{O}_{\mathrm{n}}$ the Fock state $|N\rangle$ can be completely reconstructed [in this case the corresponding entropy $S_{\mathrm{n}} = -k_{\mathrm{B}} \sum_m P_m \ln P_m$ is equal to zero]. In Fig. 6a we present a result of the numerical reconstruction of the Wigner function $W_{|n\rangle}^{(\mathrm{n})}(q,p)$ of the Fock state $|2\rangle$ for which $\langle \hat{n} \rangle = 2$ and $\langle \hat{n}^2 \rangle = 4$. Comparing Figures 5a and 6a we see that on the observation level $\mathcal{O}_{\mathrm{n}}$ the complete reconstruction of the Wigner function of the Fock state can be performed. On Fig. 6b we present a result of the numerical reconstruction on the observation level $\mathcal{O}_{\mathrm{n}}$ of the state for which $\langle \hat{n} \rangle = 2$ and $\langle \hat{n}^2 \rangle = 4.2$. This state on the observation level $\mathcal{O}_{\mathrm{n}}$ is described by the density operator (6.14) and the corresponding photon number distribution P_m is a discrete Gaussian-like function ($m \geqslant 0$). The Wigner function of this state is negative, which in particular reflects the fact that the reconstructed distribution is narrower than the Poissonian (coherent-state) photon number distribution, i.e., the state under consideration exhibits sub-Poissonian photon number distribution. To quantify the degree of the sub-Poissonian photon statistics one can utilize the Mandel Q parameter [40] defined as

$$Q = \frac{\langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2 - \langle \hat{n} \rangle}{\langle \hat{n} \rangle}, \tag{6.18}$$

which for Fock states is equal to -1 while for coherent states is equal to 0. The state is said to have sub-Poissonian photon statistics providing $Q < 0$. As seen from Figs. 6 one can (partially) reconstruct sub-Poissonian state on the observation

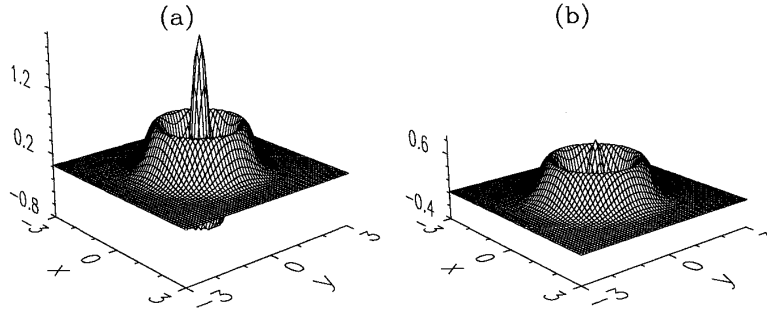

FIG. 6. The reconstructed Wigner functions of the generalized Gaussian states with $\bar{n}=2$ on the observation level $\mathcal{O}_n$: (a) the Wigner function of the Fock state $|n=2\rangle$ (in this case the Mandel Q parameter is equal to -1 and $S_n=0$); (b) the Wigner function of the state with the discrete Gaussian photon number distribution [see Eq. (6.16a)] with $\langle \hat{n}^2\rangle = 4.2$ (so that $Q=-0.9$).

level $\mathcal{O}_{\rm n}$. In addition states with the Poissonian photon statics $Q=0$ can be partially reconstructed on this observation level as well. For instance in Fig. 1h we represent a result of numerical reconstruction of the Wigner function $W^{(\rm n)}_{|\alpha\rangle}(\xi)$ of the coherent state with a Poissonian photon number distribution on the observation level $\mathcal{O}_{\rm n}$. In this case the reconstructed photon number distribution P_n [see Eq. (6.16a)] does not have a Poissonian character, and therefore the reconstructed Wigner functions of the coherent state on the observation levels $\mathcal{O}_{\rm A}$ and $\mathcal{O}_{\rm n}$ are different (compare Figs. 1b and 1h, respectively) even though the reconstructed states have the same mean photon number $\langle \hat{n}\rangle$ and the same variance σ_n^2 in the photon number distribution.

On the observation level $\mathcal{O}_{\rm n}$ we can reconstruct also the odd coherent state given by Eq. (2.46b) which is a sub-Poissonian state with the Q parameter given by the relation (we assume α to be real)

$$Q=-\frac{4\alpha^2 e^{-2\alpha^2}}{1-e^{-4\alpha^2}}=-\frac{\bar{n}}{(\cosh\alpha^2)^2}<0, \tag{6.19a}$$

where the mean photon number $\bar{n}$ in the odd coherent state is given by the relation $\bar{n}=\alpha^2\coth\alpha^2$. We have plotted the result of the numerical reconstruction of the Wigner function of the odd coherent state with $\bar{n}=2$ on the given observation level in Fig. 4g. Due to the fact, that for the given mean photon number the odd coherent state does not exhibit a significant degree of sub-Poissonian photon statistics, the corresponding Wigner function $W^{(\rm n)}_{|\alpha_o\rangle}(\xi)$ is not negative (compare Fig. 4c).

The even coherent state (2.46a) is characterized by the super-Poissonian photon statistics with the Mandel Q parameter given by the relation

$$Q=\frac{4\alpha^2 e^{-2\alpha^2}}{1-e^{-4\alpha^2}}=\frac{\bar{n}}{(\sinh\alpha^2)^2}>0, \tag{6.19b}$$

with the mean photon number given by the relation $\bar{n} = \alpha^2 \tanh \alpha^2$. From Eq. (6.19b) it follows that for large enough values of α (i.e., for large enough values of $\bar{n}$) the Mandel Q parameter is smaller than $\bar{n}$ (it tends to zero). In this case the Wigner function of the even coherent state on the observation level $\mathcal{O}_{\rm n}$ can be easily reconstructed (see Fig. 3g). We can also reconstruct on this observation level a thermal mixture for which the Mandel Q parameter is equal to $\bar{n}$ (i.e., $\langle \hat{n}^2 \rangle = 2\bar{n}^2 + \bar{n}$). In this case the Lagrange multiplier λ_2 in expression (6.14) is equal to zero and consequently the results of the reconstruction on the observation levels $\mathcal{O}_{\rm n}$ and $\mathcal{O}_{\rm th}$ (thermal observation level) are equal.

It is important to stress that all those states for which the Mandel Q parameter is less than $\bar{n}$ (in analogy with sub-Poissonian states we can call these states as the sub-thermal states) can be reconstructed on $\mathcal{O}_{\rm n}$. For all these states the Lagrange multiplier λ_2 is greater than zero and consequently the generalized partition function (6.16b) does exist. Nevertheless there are states for which $Q > \bar{n}$ (we will call these states as super-thermal states). For these state the Lagrange multiplier λ_2 is smaller than zero and $Z_{\rm n}$ given by Eq. (6.16b) is diverging. Consequently, these states cannot be reconstructed on the observation level $\mathcal{O}_{\rm n}$. In particular, the Mandel Q parameter for the squeezed vacuum state (2.41a) reads $Q = 2\bar{n} + 1$ (for $\bar{n} > 0$) and therefore we are not able to reconstruct the Wigner function of the squeezed vacuum state on $\mathcal{O}_{\rm n}$. Analogously, the even coherent state for small values of α such that $\sinh \alpha^2 < 1$ has a super-thermal photon number distribution and it cannot be reconstructed on this observation level.

The mathematical reason behind the fact that super-thermal states cannot be reconstructed on $\mathcal{O}_{\rm n}$ is closely related to the semi-infiniteness of the Fock state space of the harmonic oscillator, i.e., the photon number distribution of these states cannot be approximated by discrete Gaussian distributions P_m (6.16a) on the interval $m \in [0, \infty)$. In principle, there exist two ways how to regularize the problem: one can either expand the Fock space and to introduce "negative" Fock states, i.e., $m \in (-\infty, \infty)$. Alternatively, one can assume finite-dimensional Fock space such that $m \in [0, s]$. In both these cases $Z_{\rm n}$ for super-thermal states is finite and in principle $\hat{\sigma}_{\rm n}$ can be reconstructed. Obviously, in this case the relevance of reconstructed Wigner functions to a real situation is delicate and we are not going to discuss this problem in the present paper.

6.3. *Reconstruction of Wigner Functions and Marcinkiewicz Theorem*

From the generalization of the Marcinkiewicz theorem [41] due to Rajagopal and Sudarshan [42] it follows that *a quasiprobability with a finite-number cumulant expansion has to have a Gaussian characteristic function.* In other words, the quantum-mechanical state is either characterized by the first- and second-order cumulants or by an infinite number of nonzero cumulants. From here it follows that within the framework of the phase-space formalism in the (q, p)-phase space we can divide all quantum-mechanical states into three groups:

1. *Gaussian states* are the states with the Gaussian Wigner function $W(q, p)$ of the form (4.21). These states (such as coherent states or squeezed coherent states) are completely characterized by mean values of the operators $\hat{q}$ and $\hat{p}$ and the variances $\langle (\Delta\hat{q})^2 \rangle$, $\langle (\Delta\hat{p})^2 \rangle$ and $\langle \{ \Delta\hat{q}\ \Delta\hat{p} \} \rangle$. Characteristic functions of these functions can be expressed as exponentials of polynomials of the order less or equal to 2 [compare with Eq. (2.34)].

2. *Generalized Gaussian states* are the states for which Wigner functions $W(q, p)$ in the (q, p)-phase space are not Gaussian functions but they are described by density operators that can be expressed as exponentials of polynomials of the order less or equal to two in terms of any observables, i.e., these states are characterized just by first two cumulants in terms of given observables (all higher-order cumulants are equal to zero). Very good example of the generalized Gaussian states are the Fock states. These states have non-Gaussian Wigner functions [see Eq. (2.37b)], but they are completely described by the density operator (6.14) such that $\langle \hat{n}^2 \rangle = \langle \hat{n} \rangle^2$. As a consequence of this relation it follows that the higher-order moments of the operators $\hat{q}$ and $\hat{p}$ in the Fock state $|n\rangle$ can be expressed in terms of the mean values $\langle \hat{q}^2 \rangle$ and $\langle \hat{p}^2 \rangle$ [see Eq. (2.38)]. As we said, generalized Gaussian states are described in the (q, p)-phase space by non-Gaussian Wigner function. Nevertheless, in an appropriate phase space they can be described by Gaussian functions. To be more specific, one can consider a state described by the density operator $\hat{\rho} \simeq \exp[-P_2(\hat{\phi}, \hat{n})]$ where $P_2(\hat{\phi}, \hat{n})$ is a polynomial of the second-order in terms of the number $(\hat{n})$ and phase $(\hat{\phi})$ operators. This state has a Gaussian Wigner function $W(n, \phi)$ in the (n, ϕ)-phase space [43] (see also Ref. [9]) but is described by non-Gaussian Wigner function in the (q, p)-phase space. This is because there does not exist a linear transformation from one phase space to the other (which obviously is dictated by the relations between corresponding observables). It is clear, that there are observation levels on which generalized Gaussian can be completely reconstructed via the measurement of just small number of observables.

3. *Non-Gaussian states* are characterized by an infinite number of nonzero cumulants of arbitrary observables. They are described by non-Gaussian Wigner functions and to perform their complete reconstruction one has to perform a measurement of an infinite number of independent moments. Even and odd coherent states can serve as examples of non-Gaussian states. These states can be completely reconstructed only on the observation level $\mathcal{O}_0$ when an infinite number of independent moments of observables is measured. The partial reconstruction of non-Gaussian states on reduced observation levels can be performed. This partial reconstruction corresponds to a particular truncation scheme [26] and the effectiveness of the truncation can be quantified with the help of the corresponding entropy. In Fig. 7 we present numerical values (in a form of a histogram) of entropies on the phase-insensitive (Fig. 7a) and the phase-sensitive (Fig. 7b) observation levels for various quantum-mechanical states of light (we assume that all these states are characterized by a mean photon number $\bar{n} = 2$). Comparing

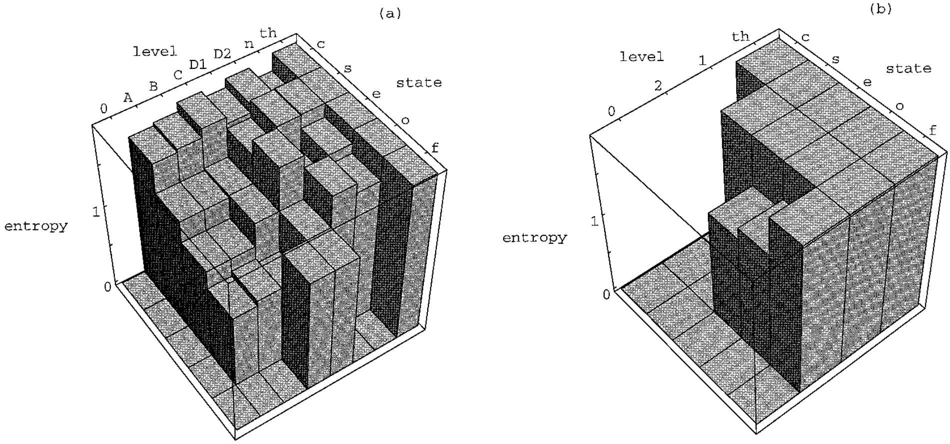

FIG. 7. We present numerical values of entropies on phase insensitive (a) and phase sensitive (b) observation levels for the following quantum-mechanical states of light: the coherent state (denoted as (c); the squeezed vacuum state (s); the even coherent state (e); the odd coherent state (o), and the Fock state (f). The mean photon number is equal to 2. The observation levels $\mathcal{O}_X$ are denoted as X.

entropies corresponding to different observation levels for a given state we can determine how good or bad the reconstruction is (or, in other words, how good or bad the applied truncation scheme is).

6.4. *Optical Homodyne Tomography and MaxEnt Principle*

From the point of view of the formalism presented in this paper it follows that from the probability density distribution $W_{|\Psi\rangle}(x_\theta)$ [see Eq. (2.21a)] which corresponds to a measurement of all moments $\langle \hat{x}_\theta^n \rangle$, the generalized canonical density operators $\hat{\sigma}_{x_\theta}$ [see also Eq. (6.3)]

$$\hat{\sigma}_{x_\theta} = \frac{1}{Z_{x_\theta}} \exp\left[-\int_{-\infty}^{\infty} dx_\theta \, |x_\theta\rangle\langle x_\theta| \, \lambda(x_\theta) \right] \tag{6.20}$$

can be constructed. The Lagrange multipliers $\lambda(x_\theta)$ are given by an infinite set of equations

$$W_{|\Psi\rangle}(x_\theta) = \sqrt{2\pi\hbar} \, \langle x_\theta | \, \hat{\sigma}_{x_\theta} \, | x_\theta \rangle; \qquad \forall x_\theta \in (-\infty, \infty). \tag{6.21}$$

If probability distributions $W_{|\Psi\rangle}(x_\theta)$ for all values of $\theta \in [0, \pi]$ are known then the density operator on the complete observation level can be obtained in the form

$$\hat{\rho}_0 = \frac{1}{Z_0} \exp\left[-\int_0^\pi d\theta \int_{-\infty}^{\infty} dx_\theta \, |x_\theta\rangle\langle x_\theta| \, \lambda(x_\theta) \right] \tag{6.22}$$

and the corresponding Wigner function can be reconstructed. The optical homodyne tomography can be understood as a method how to find a relation between measured distributions $W_{|\Psi\rangle}(x_\theta)$ and the Lagrange multipliers $\lambda(x_\theta)$ for all values of x_θ and θ. As we have shown earlier in this section, the Gaussian and the generalized Gaussian states can be completely reconstructed on reduced observation levels based on a measurement of just finite number of moments of system observables, and therefore the optical homodyne tomography is essentially not needed as a method for reconstruction of Wigner functions in these cases. On the other hand, the non-Gaussian states can, in principle, be reconstructed, but in practice the reconstruction of their Wigner functions is associated with a measurement of an infinite number of independent moments of system observables which is not realistic. In the experiments by Raymer *et al.* [17] only a finite number of values of θ have been considered, i.e., these types of experiments are associated with observation level for which the corresponding generalized canonical density operator reads

$$\hat{\sigma} = \frac{1}{Z} \exp\left[-\sum_j \int_{-\infty}^{\infty} dx_{\theta_j} \, |x_{\theta_j}\rangle\langle x_{\theta_j}| \, \lambda(x_{\theta_j}) \right]. \tag{6.23}$$

Usually this kind of measurement results into a very good reconstruction of Wigner functions (such that the corresponding entropy is close to zero for pure states). Nevertheless, a certain attention has to be paid for highly squeezed states, such as the Vogel-Schleich phase states [44], for which the measurement of distributions $W_{|\Psi\rangle}(x_{\theta_j})$ can be problematic. Namely, $W_{|\Psi\rangle}(x_{\theta_j})$ can be very "wide", so that the normalization condition is not fulfilled in a domain of physically accessible values of x_{θ_j}.

7. Conclusions

We have presented a universal method for a reconstruction of Wigner functions of quantum-mechanical states of light. This method allows us to reconstruct Wigner functions with a certain degree of credibility (quantified with the help of entropies) from a set of measured values of system observables. This set of observables defines a given observation level. We have to stress that the concept of observation levels plays very important role in our attempt to measure and understand nonclassical effects of quantum states of light. In fact, a measurement of second-order quadrature squeezing is implicitly associated with the observation level $\mathcal{O}_2$. Analogously, a measurement of the Mandel Q-parameter is associated with the observation level $\mathcal{O}_\mathrm{n}$. We know that a reduction of quantum fluctuations (i.e., quadrature squeezing, sub-Poissonian photon statistics, etc.) has its origin in quantum interference between coherent components of superposition states of light. In the following paper we will utilize the concept of observation levels and corresponding entropies to study the decay of quantum coherences between coherent components of superposition states. We will also show how to introduce entropic uncertainty relations on a particular observation level.

Appendix A: Does There Exist Observation Level $\mathcal{O}_{\rm N} \equiv \{\hat{P}_N = |N\rangle\langle N|\}$?

In this Appendix we assume that the observable $\hat{G}_\nu$ is chosen to be the projector $\hat{P}_N = |N\rangle\langle N|$. We assume that the only information about a measured system, which is prepared in a state described by the density operator $\hat{\rho}$, is the expectation value $P_N = \mathrm{Tr}[\hat{\rho}\hat{P}_N]$. In this case the generalized canonical density operator $\hat{\sigma}_{\rm N}$ reads

$$\hat{\sigma}_{\rm N} = \frac{\exp[-\lambda\hat{P}_N]}{\mathrm{Tr}\{\exp[-\lambda\hat{P}_N]\}}. \tag{A.1}$$

The Lagrange multiplier λ has to be evaluated from the relation $\mathrm{Tr}[\hat{\sigma}_{\rm N}\hat{P}_N] = \mathrm{Tr}[\hat{\rho}\hat{P}_N] = P_N$. Before we evaluate the explicit expression for λ as a function of P_N, we turn our attention to the normalization factor (generalized partition function) $Z_{\rm N} = \mathrm{Tr}\exp[-\lambda\hat{P}_N]$. Using the relation $(\hat{P}_N)^2 = \hat{P}_N$ we can write the partition function as

$$Z_{\rm N} = \mathrm{Tr}[1 - \hat{P}_N + e^{-\lambda}\hat{P}_N] = e^{-\lambda} - 1 + \sum_{k=0}^{\infty} 1, \tag{A.2}$$

which means that $Z_{\rm N}$ diverges. To overcome this problem we regularize the partition function for a while and instead of an infinite Hilbert space formed out of all Fock states we assume an $(s+1)$ dimensional Hilbert space formed out of Fock states $|0\rangle, \ldots, |s\rangle$ (here we assume $s \gg N$). We can now write the required normalization factor as ($Z_{\rm N}$ and λ depend now on the normalization procedure and therefore we will label them with superscript s)

$$Z_{\rm N}^{(s)} = e^{-\lambda^{(s)}} - 1 + \sum_{k=0}^{s} 1 = e^{-\lambda^{(s)}} + s. \tag{A.3}$$

In this regularized case we can find the expression for the parameter $\lambda^{(s)}$ as

$$e^{-\lambda^{(s)}} = \frac{sP_N}{1 - P_N}. \tag{A.4}$$

If we insert expressions (A.3) and (A.4) into a definition of the generalized canonical density operator (A.1) we find

$$\hat{\sigma}_{\rm N}^{(s)} = P_N\,|N\rangle\langle N| + \frac{1 - P_N}{s}\sum_{k \neq N}^{s} |k\rangle\langle k| = \sum_{m=0}^{s} P_k^{(s)}\,|k\rangle\langle k|, \tag{A.5}$$

where $P_k^{(s)} = (1 - P_N)/s$ for $k \neq N$. This expression has an attractive interpretation. We know that the mean value of the measured observable is P_N. On the other hand, we have *no* knowledge about mean values of other operators $|k\rangle\langle k|$. Therefore, following the MaxEnt principle we have to assume that these observables have the *same* probability, i.e., $P_k^{(s)} = (1 - P_N)/s$ where $1 - P_N$ is the *total* probability of unknown mean values and s is the number of "unmeasured" observables $|k\rangle\langle k|$. Entropy corresponding to the generalized canonical density operator $\hat{\sigma}_{\mathrm{N}}^{(s)}$ reads

$$S_{\mathrm{N}}^{(s)} = -k_{\mathrm{B}} \,\mathrm{Tr}[\hat{\sigma}_{\mathrm{N}}^{(s)} \ln \hat{\sigma}_{\mathrm{N}}^{(s)}] = -k_{\mathrm{B}} \sum_{k=0}^{s} P_k^{(s)} \ln P_k^{(s)}$$
$$= -k_{\mathrm{B}}[P_N \ln P_N + (1 - P_N) \ln (1 - P_N)] + k_{\mathrm{B}}(1 - P_N) \ln s. \tag{A.6}$$

From the last expression it follows that in the limit $s \to \infty$ we have

$$\lim_{s \to \infty} S_{\mathrm{N}}^{(s)} = \begin{cases} 0 & \text{if} \quad P_N = 1, \\ \infty & \text{if} \quad P_N < 1. \end{cases} \tag{A.7}$$

Also the mean photon number in the state given by $\hat{\sigma}_{\mathrm{N}}^{(s)}$ diverges in the limit $s \to \infty$. Namely,

$$\langle \hat{n} \rangle^{(s)} = \mathrm{Tr}\,[\hat{n}\hat{\sigma}_{\mathrm{N}}^{(s)}] = NP_N - \frac{1 - P_N}{s} + \frac{1 - P_N}{2}(s + 1), \tag{A.8}$$

which in the limit $s \to \infty$ reads

$$\lim_{s \to \infty} \langle \hat{n} \rangle^{(s)} = \begin{cases} N & \text{if} \quad P_N = 1, \\ \infty & \text{if} \quad P_N < 1. \end{cases} \tag{A.9}$$

We can conclude that if an incomplete measurement of the photon number distribution, such that $\sum_n P_n < 1$, is performed then the Wigner function cannot be reconstructed if the mean photon is not known.

Appendix B: Phase-Space Representation of the Eigenstate of the Position Operator

We represent the position state $|\bar{q}\rangle$ (i.e., the eigenstate of the position operator $\hat{q}$) as the displaced squeezed state $|\bar{q}, r\rangle \equiv \hat{D}(\bar{q}, 0)\, \hat{U}(\pi/2)\, \hat{S}(r)\, |0\rangle$ [where the operators $\hat{D}(\bar{q})$, $\hat{U}(\pi/2)$, and $\hat{S}(r)$ are given by Eqs. (2.19), (2.23) and (2.41b), respectively], in the limit of infinite squeezing:

$$|\bar{q}\rangle = \lim_{r \to \infty} \hat{D}(\bar{q}, 0)\, \hat{U}(\pi/2)\, \hat{S}(r)\, |0\rangle. \tag{B.1}$$

The action of the position operator $\hat{q}$ on the state $|\bar{q}, r\rangle$ is

$$\hat{q}\, |\bar{q}, r\rangle = \bar{q}\, |\bar{q}, r\rangle + e^{-r} \sqrt{\frac{\hbar}{2}}\, \hat{D}(\bar{q}, 0)\, \hat{U}(\pi/2)\, \hat{S}(r)\, |1\rangle, \tag{B.2}$$

where $|1\rangle$ is a Fock state with one excitation quantum. In the limit $r \to \infty$ we formally obtain from (B.2) the eigenvalue equation for the position operator. The Wigner function of the state $|\bar{q}, r\rangle$ is given by the expression

$$W_{|\bar{q}, r\rangle}(q, p) = \frac{1}{\sigma_q \sigma_p} \exp\left[-\frac{1}{2\hbar} \frac{(q-\bar{q})^2}{\sigma_q^2} - \frac{1}{2\hbar} \frac{p^2}{\sigma_p^2} \right], \tag{B.3}$$

where $\sigma_q^2 = \exp(-2r)/2$ and $\sigma_p^2 = \exp(2r)/2$. The Wigner function (B.3) in the limit $r \to \infty$ can be understood as the Wigner function of the position state $|\bar{q}\rangle$. Using the general formalism [see Sec. 2] we can obtain from Eq. (B.3) the expression for the marginal distribution $W_{|\bar{q}, r\rangle}(q)$:

$$W_{|\bar{q}, r\rangle}(q) = \frac{1}{\sigma_q} \exp\left[-\frac{1}{2\hbar} \frac{(q-\bar{q})^2}{\sigma_q^2} \right], \tag{B.4}$$

with the help of which we can evaluate mean values of all powers of the position operator in the state $|\bar{q}, r\rangle$

$$\langle \hat{q}^n \rangle = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dq\, q^n W_{|\bar{q}, r\rangle}(q) = \sum_{m=0}^{[n/2]} \binom{n}{2m} (\bar{q})^{n-2m} (2m-1)!!\, (\hbar\sigma_q^2)^m, \tag{B.5}$$

where $[x]$ denotes the largest integer smaller than x. From Eq. (B.5) it is obvious that in the limit of infinite squeezing we have $\langle \bar{q}, r|\, \hat{q}^n\, |\bar{q}, r\rangle \to (\bar{q})^n$. Taking into account the relation (6.1) we can conclude that the state $|\bar{q}, r\rangle$ in the limit $r \to \infty$ is equal to the eigenstate $|\bar{q}\rangle$ of the position operator $\hat{q}$.

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