A SHORT INTRODUCTION TO QUANTUM ENTANGLEMENT

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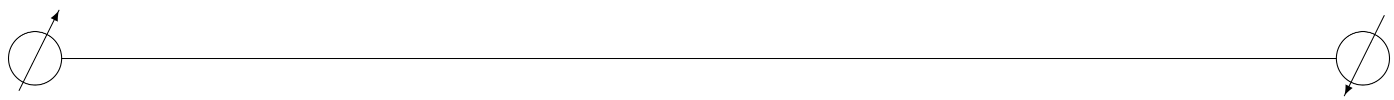
Historical background

- entanglement - a relationship or involvement that compromises the participants
- quantum entanglement - introduced by E.Schrödinger
  ("entanglement of predictions")

  E. Schrödinger, *Die gegenwärtige Situation in der Quantenmechanik*
  Naturwissenschaften 23: pp.807-812; 823-828; 844-849 (1935)
  `http://www.tu-harburg.de/rzt/rzt/it/QM/cat.html`
- existence of two-particle states $\Psi_{AB} \neq \phi_A \otimes \chi_B$
- properties of individual systems seems to be senseless in such cases
- strange "correlations" of predictions between experiments on individual particles

Einstein-Podolski-Rosen problem

- realism $=$ ability of deterministic predictions require that the state possess the property before the measurement, i.e. even without the measurement
- locality $=$ no instantenuous actions, i.e. operations on system A does not affect the properties of system B instantenuously, and vice versa
- EPR requirement every theory must satisfy such conditions
- two half-spins in state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A \otimes |\uparrow\rangle_B)$



- fact: measuring $\vec{a}\cdot\vec{\sigma}\otimes I_B$ determines outcomes of $I_A \otimes \vec{b}\cdot\vec{\sigma}$ with certainty if $\vec{b}=\vec{a}$
- local realism $\Rightarrow$ spin B must possess the property "having spin $\vec{a}$" before the measurement, or we must consider existence of instantenuous nonlocal action

Einstein-Podolski-Rosen problem

- two half-spins in state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A \otimes |\uparrow\rangle_B)$
- local realism $\Rightarrow$ spin B must possess the property "having spin $\vec{a}$" before the measurement, or we must consider existence of instantenuous nonlocal action
- 1st BUT: choice of $\vec{a}$ is arbitrary and can be decided after the state is created
  local realism $\Rightarrow$ spin is determined in all directions
- 2nd BUT: QT description $\Rightarrow$ spin can be determined at most in one direction !!!
- EPR conclusion $\longrightarrow$ quantum state description is incomplete and allows spooky actions at a distance
- alternative: local hidden variables predicting individual outcomes
- EPR believed that such theory is possible

Bell inequalities

- local realistic model: $A(\vec{a},\lambda), B(\vec{b},\lambda) \in \pm 1$ and $\langle \vec{a} \otimes \vec{b} \rangle = \int \mathrm{d}\lambda \varrho(\lambda) A(\vec{a},\lambda) B(\vec{b},\lambda)$
- λ is the hidden parameter, or set of parameters
- knowledge of $\lambda \Rightarrow$ ability to make deterministic predictions for all measurements
- local hidden variable model

$$\begin{aligned} \mathcal{B}_{LHV} &= |\langle \vec{a} \otimes (\vec{b} + \vec{b}') + \vec{a}' \otimes (\vec{b} - \vec{b}') \rangle| \\ &= \left| \int \mathrm{d}\lambda A(\vec{a},\lambda)[B(\vec{b},\lambda) + B'(\vec{b}',\lambda)] + A'(\vec{a}',\lambda)[B(\vec{b},\lambda) - B'(\vec{b}',\lambda)] \right| \\ &\leq \int \mathrm{d}\lambda \left| A(\vec{a},\lambda)[B(\vec{b},\lambda) + B'(\vec{b}',\lambda)] + A'(\vec{a}',\lambda)[B(\vec{b},\lambda) - B'(\vec{b}',\lambda)] \right| \\ &\leq 2 \end{aligned} \tag{1}$$

- quantum theory prediction for singlet

$$\begin{aligned} \mathcal{B}_{QM} &= |\langle \vec{a} \otimes (\vec{b} + \vec{b}') + \vec{a}' \otimes (\vec{b} - \vec{b}') \rangle| = |-\vec{a} \cdot (\vec{b} + \vec{b}') - \vec{a} \cdot (\vec{b} - \vec{b}')| \\ &= 2\sqrt{2} > 2 \geq \mathcal{B}_{LHV} \end{aligned} \tag{2}$$

- QM violates the LHV model contraints given by Bell inequality

Outline

1. History and motivation
2. LOCC operations and entanglement
3. Maximally entangled states
4. Applications of maximally entangled states

Pure states entanglement

- entanglement: difference between classical and quantum
  - feature of quantum state necessary in violation of BI, nonexistence of LHV model
- definition: pure state $|\Phi\rangle_{AB}$ is entangled if and only if $|\Phi\rangle_{AB} \neq |\phi\rangle_A \otimes |\chi\rangle_B$
- Schmidt decomposition: (important tool)

$$|\Phi\rangle_{AB} = \Sigma_{j=0}^{d-1} \sqrt{\lambda_j} |e_j\rangle_A \otimes |f_j\rangle_B \tag{3}$$

  where $\langle e_j | e'_j \rangle = \delta_{jj'}$, $\langle f_j | f'_j \rangle = \delta_{jj'}$ and λ_j are positive and sum up to unity. Hence all states are locally unitary equivalent to states $|\Psi\rangle_{AB} = a|00\rangle + b|11\rangle = (U_A \otimes U_B)|\Phi\rangle_{AB}$.
- $\vec{\lambda}_\Phi = (\lambda_0, \ldots, \lambda_{d-1})$ is the vector of Schmidt numbers ordered decreasingly, i.e. $\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{d-1}$.
- what about mixed states?

Concept of LOCC operations

- central notion describing specific manipulation of physical systems
- LOCC $=$ local operations (local measurements, local Hamiltonians) and classical communication
- for classical states:
  - all states are closed under LOCC operations, i.e. for all probability distributions $\pi(a,b), \pi'(a,b)$: $\pi \leftrightarrow \pi'$ by means of LOCC
  - all classical operations are LOCC type
- for pure quantum states:
  - factorized states are closed under LOCC operations
  - entangled pure states can be transformed into factorized states
- LOCC-based partial ordering

$$\varrho \succ \omega \text{ if there exists } \mathcal{E}_{\mathrm{LOCC}} \text{ such that } \mathcal{E}_{\mathrm{LOCC}}[\varrho] = \omega$$

Entanglement for mixed states

- LOCC-based partial ordering: $\varrho \succ \omega$ if there exists $\mathcal{E}_{\rm LOCC}$ such that $\mathcal{E}_{\rm LOCC}[\varrho] = \omega$
- separable states $\mathcal{S}_{\rm sep}$
  - def 1: set of LOCC-smallest states
  - def 2: convex hull of factorized states, i.e. $\varrho = \Sigma_j p_j |\phi_j\rangle\langle\phi_j| \otimes |\chi_j\rangle\langle\chi_j|$.
  - closed under LOCC operations
  - every state can be transformed into arbitrary separable state
- entangled states: complement of the set of separable states, i.e. $\mathcal{S}_{\rm ent} = \mathcal{S}(\mathcal{H}) \setminus \mathcal{S}_{\rm sep}$
- formal definition: a state ϱ is entangled if and only if it cannot be written in the form

$$\varrho \neq \Sigma_j p_j |\phi_j\rangle\langle\phi_j| \otimes |\chi_j\rangle\langle\chi_j|$$

Maximally entangled states

- definition: states from which all states can be prepared by deterministic LOCC
- alternatively, largest element(s) with respect to LOCC ordering
- is/are there such state/states? if yes, are they LOCC related?
- sufficient to prove for pure states, because mixed states are just classical distributions over pure states, i.e. can be prepared by means of LOCC

Maximally entangled pure states

- definition: states from which all states can be prepared by deterministic LOCC
- pure states: $|\Psi\rangle \to |\Phi\rangle$ iff $\vec{\lambda}_\Psi \prec \vec{\lambda}_\Phi$ (majorization criterion), i.e. $\Sigma_{j=0}^{J}\lambda_j^\Psi \leq \Sigma_{j=0}^{J}\lambda_j^\Phi$ for all $J = 0, \ldots, d-1$.
- maximally entagled pure state $\lambda_j^\Psi = 1/d$ for all j, i.e. $|\Psi_+\rangle = \frac{1}{\sqrt{d}}\Sigma|j\rangle_A \otimes |j\rangle_B$.
- preparation of $|\Psi\rangle = a|00\rangle + b|11\rangle$:
  1. addition of ancilla $|0\rangle_{A'} \otimes |\Psi_+\rangle_{AB}$
  2. local unitary operation $|00\rangle_{AA'} \to a|00\rangle_{AA'} + b|11\rangle_{AA'}$, $|01\rangle_{AA'} \to b|01\rangle_{AA'} + a|10\rangle_{AA'}$ resulting in state $\frac{1}{\sqrt{2}}[|0\rangle_{A'} \otimes (a|00\rangle_{AB} + b|11\rangle_{AB}) + |1\rangle_{A'} \otimes (b|10\rangle_{AB} + a|01\rangle_{AB})]$
  3. measurement $|0\rangle\langle 0|_{A'} \otimes I_{AB} - |1\rangle\langle 1|_{A'} \otimes I_{AB} = \sigma_z^{A'} \otimes I_{AB}$
  4. Alice sends result to Bob
  5. Bob performs $\sigma_0 = I$, or $\sigma_1 = \sigma_x$ on his qubit to end up with state $a|00\rangle + b|11\rangle$ deterministically.

Maximally entangled states

- solution & definition: state is maximally entangled iff it is pure and its subsystems are in total mixture state, i.e. $\mathrm{Tr}_B \Psi_{AB} = \mathrm{Tr}_A \Psi_{AB} = \frac{1}{2} I$.
- LOCC transformations can transform maximally entangled state to arbitrary other state
- All maximally entangled states Ψ, Ψ' are locally unitarily equivalent, i.e.
  $\Psi' = (U_A \otimes U_B)\Psi$, in fact $\Psi' = (U_A \otimes I)\Psi$
- Maximally entangled state cannot be prepared from any other state by mens of LOCC operations, i.e. $\varrho \not\to \Psi_+$
- if $\varrho \to \Psi_+$, then ϱ is maximally entangled state.

Application: superdense coding

- situation: Alice and Bob share $|\Psi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
- step 1 (encoding): apply operation $\sigma_j \otimes I_B$ on state $|\Psi_+\rangle$
- main trick: orthogonal basis related by local unitary transformations

$$\langle(\sigma_k \otimes I_B)\Psi_+|(\sigma_j \otimes I_B)\Psi_+\rangle = \frac{1}{2}\mathrm{Tr}[\sigma_j\sigma_k] = \delta_{jk} \tag{4}$$

- step 2 (qubit transfer): Alice sends her qubit to Bob
- step 3 (measurement): Bell measurement in basis $|(\sigma_j \otimes I)\Psi_+\rangle$ gives j
- usual magic note: qubit channel transfers 2 classical bits per one usage, but at most single bit can be extracted from single qubit alone [classical bound]
- transfer is secure, because the transmitted qubit does not contain any information
- 2cbits=qbit + EPR

Application: quantum teleportation

- not a matter transfer and not instantenuous = not StarTrek teleportation
- mathematics behind

$$\begin{aligned} |\phi\rangle_S \otimes |\Psi_+\rangle_{AB} &= \frac{1}{\sqrt{2}}|\phi\rangle_S \otimes [|\uparrow\rangle_A|\uparrow\rangle_B + |\downarrow\rangle_A|\downarrow\rangle_B] \\ &= \frac{1}{2}\Sigma_{j=0}^{3}|(I_S \otimes \sigma_j)\Psi_+\rangle_{SA} \otimes |\sigma_j\phi\rangle_B \end{aligned} \tag{5}$$

- mutually orthogonal states $|(\sigma_j \otimes I)\Psi_+\rangle$ forming Bell measurement
- step 1 (measurement): Alice measures outcome j and Bob's spin is in state $|\sigma_j\phi\rangle$
- step 2 (communication): transfer of 2 bits of information encoding the value j
- step 3 (correction): Bob applies σ_j to recover the original state $|\phi\rangle_B$ ($\sigma_j^2 = I$)
- note: teleportation transfers "only" (quantum) information and it is not instantenuous
- qbit=2cbits+EPR

Entanglement theory

- decide (theoretically/experimentally) whether a given state is entangled, or not
- task: entanglement identification and quantification (entanglement measures)
- lacking of operational meaning of entanglement
- Bell inequality? $\longrightarrow$ there are (mixed) entangled states with LHV models (Werner,1982)
- teleportation? $\longrightarrow$ existence of bound entangled states
- superdense coding? $\longrightarrow$ entangled states with $C_{\mathrm{quantum}}(\varrho) \leq C_{\mathrm{class}}^{\mathrm{max}}$
- correlations? $\longrightarrow$ equivalent for pure states, but for mixed states the intrinsic quantum correlations (entanglement) cannot be separated from "classical" correlations
- nonlocality $\not\Leftrightarrow$ entanglement $\not\Leftrightarrow$ correlations

Concluding remarks

- state ϱ is entangled if and only if $\varrho \neq \Sigma_j p_j |\psi_j\rangle\langle\psi_j| \otimes |\phi_j\rangle\langle\phi_j|$
- main concept: LOCC operations and LOCC-induced ordering
- nonlocality $\not\Leftrightarrow$ entanglement $\not\Leftrightarrow$ nonclassical correlations
- applications: teleportation, superdense coding, cryptography, q-computation
- entanglement is still not conceptually understood (lacking of operational definition)
- easy for pure states and two qubits
- multipartite entanglement (phase transitions, monogamy)

Good references

1. Ryszard Horodecki, Pawel Horodecki, Michal Horodecki and Karol Horodecki: Quantum entanglement, Rev.Mod.Phys, quant-ph/0702225
2. Martin Plenio, Shashank Virmani: An intorduction to entanglement measures, quant-ph/0504163
3. Aditi Sen(De), Ujjwal Sen, Maciej Lewenstein, Anna Sanpera: Lectures on Quantum Information: Chapter 1 (The separability versus entanglement problem), quant-ph/0508032
4. Karol Zyczkowski and Ingemar Bengtsson: An introduction to quantum entanglement: a geometric approach, quant-ph/0606228

Entanglement measures - axioms

1. Sharpness $E(\varrho) = 0$ iff ϱ is not entangled
2. Local unitary invariance $E(\varrho) = E(U_1 \otimes U_2 \varrho U_1^\dagger U_2^\dagger)$
3. Nonincresing under LOCC $E(\varrho) \geq \Sigma_j p_j E(\mathcal{M}_j[\varrho])$
4. Normalization $E(\varrho)$ is maximal only for maximally entangled states
5. Convexity $E(\varrho) \leq \Sigma_j p_j E(\varrho_j)$
6. Additivity $E(\varrho \otimes \sigma) = E(\varrho) + S(\varrho)$
7. Continuity