Multisymplectic Geometry in Classical Field Theory

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Introduction

- Canonical quantization is based on Hamilton mechanics
- Classical field theory on 1D timespace is just mechanics.
- DeDonder, Weyl, Kanatchikov, ...

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- Generalized coordinates q^a
- 1D time t
- Lagrangian $L(q, \dot{q})$
- Extended configuration space $N = Q \times \mathbb{R}$
- Extremal action principle $S[q(t)] = \int L dt$

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