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Graded contraction of the Gell-Mann graded sl(3,C)

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Group gradings of Lie algebras

- $\mathcal{L} \ldots$ Lie algebra over $\mathbb{C}$, finite–dimensional, simple

- decomposition $\mathcal{L} = \bigoplus_{i \in G} \mathcal{L}_i$ is a **group grading** of $\mathcal{L}$, if $[\mathcal{L}_i, \mathcal{L}_j] \subseteq \mathcal{L}_{i+j}$, where G is an Abelian group

- **fine grading**, if subspaces $\mathcal{L}_i$ are minimal

- **finest grading**, if $\dim(\mathcal{L}_i) = 1$

- symmetry of a grading: $\operatorname{Aut}\Gamma : \ g \in \operatorname{Aut}\mathcal{L}$

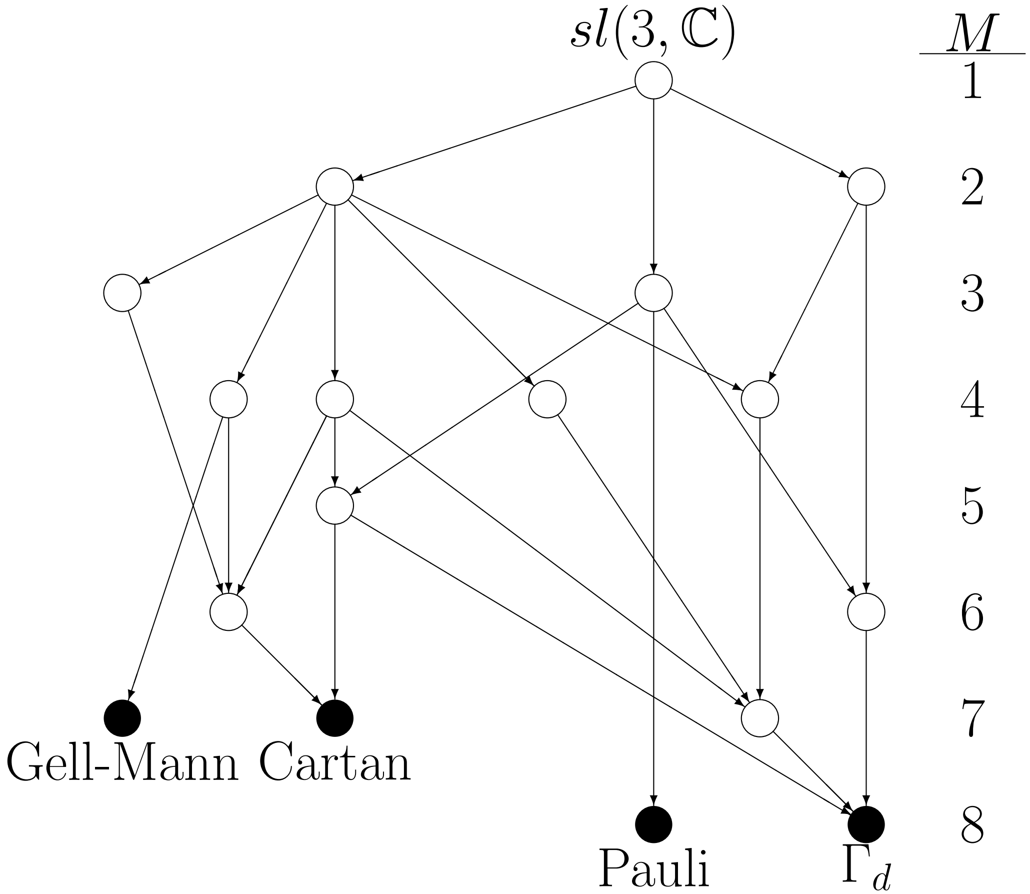
$$g\mathcal{L}_i = \mathcal{L}_{\pi_g(i)},$$

  where π_g is permutation on G

- $\operatorname{Stab}\Gamma = \{g \in \operatorname{Aut}\mathcal{L} \mid g\mathcal{L}_i = \mathcal{L}_i \ \forall i \in G\}$

- **symmetry group $\boldsymbol{SG}$** of the grading is $\operatorname{Aut}\Gamma / \operatorname{Stab}\Gamma$

The hierarchy of 17 group gradings of $sl(3,\mathbb{C})$



Fine gradings of $sl(3,\mathbb{C})$

- Gell-Mann $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ $\quad |SG| = 24 \quad sl(3,\mathbb{C}) = \mathcal{L}_{001} \oplus \mathcal{L}_{111} \oplus \mathcal{L}_{101} \oplus \mathcal{L}_{011} \oplus \mathcal{L}_{110} \oplus \mathcal{L}_{010} \oplus \mathcal{L}_{100}$

$$= \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -a-b \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

- Cartan $\mathbb{Z}_3 \times \mathbb{Z}_3$ $\quad |SG| = 12 \quad sl(3,\mathbb{C}) = \mathcal{L}_{00} \oplus \mathcal{L}_{10} \oplus \mathcal{L}_{01} \oplus \mathcal{L}_{11} \oplus \mathcal{L}_{-1-1} \oplus \mathcal{L}_{0-1} \oplus \mathcal{L}_{-10}$

$$= \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -a-b \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- Pauli $\mathbb{Z}_3 \times \mathbb{Z}_3$ $\quad |SG| = 48 \quad sl(3,\mathbb{C}) = \mathcal{L}_{01} \oplus \mathcal{L}_{02} \oplus \mathcal{L}_{10} \oplus \mathcal{L}_{20} \oplus \mathcal{L}_{11} \oplus \mathcal{L}_{22} \oplus \mathcal{L}_{12} \oplus \mathcal{L}_{21}$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega^2 \\ 1 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & \omega \\ 1 & 0 & 0 \\ 0 & \omega^2 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & \omega^2 & 0 \\ 0 & 0 & \omega \\ 1 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & \omega^2 \\ 1 & 0 & 0 \\ 0 & \omega & 0 \end{pmatrix}$$

where $\omega = e^{\frac{2\pi i}{3}}$

- $\mathbb{Z}_8$–grading Γ_d $\quad |SG| = 4 \quad sl(3,\mathbb{C}) = \mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3 \oplus \mathcal{L}_4 \oplus \mathcal{L}_5 \oplus \mathcal{L}_6 \oplus \mathcal{L}_7$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Graded contractions of Lie algebras

- we define on graded $\mathcal{L}$ a new product $[\,,\,]_\varepsilon$ by

$$[x,y]_\varepsilon = \varepsilon_{ij}[x,y]$$

  for $x \in \mathcal{L}_i,\ y \in \mathcal{L}_j$ and $\varepsilon_{i,j} \in \mathbb{C}$

- if $\mathcal{L}^\varepsilon = (\mathcal{L}, [\,,\,]_\varepsilon)$ is a Lie algebra i.e. for all $i,j,k \in G$
  - $\varepsilon_{ij} = \varepsilon_{ji}$
  - $[x,[y,z]_\varepsilon]_\varepsilon + [z,[x,y]_\varepsilon]_\varepsilon + [y,[z,x]_\varepsilon]_\varepsilon = 0 \qquad \forall x \in \mathcal{L}_i, \forall y \in \mathcal{L}_j, \forall z \in \mathcal{L}_k$

  hold, then $\mathcal{L}^\varepsilon$ is a **graded contraction** of $\mathcal{L}$

- $\varepsilon = (\varepsilon_{ij}) \ldots$ contraction matrix, $\mathcal{S} \ldots$ system of contraction equations,
  $\mathcal{R}(\mathcal{S}) \ldots$ set of all solutions of $\mathcal{S}$

- **normalization matrix** $\alpha := (\alpha_{ij})$, where

$$\alpha_{ij} = \frac{a_i a_j}{a_{i+j}} \quad \text{for } i,j \in G$$

  and $a_i \in \mathbb{C} \setminus \{0\}$ for $i \in G$

Symmetry of solutions

$$\boxed{SG \text{ of the grading}} \cong \boxed{SG \text{ of contraction equations } \mathcal{S}} \cong \boxed{SG \text{ of solutions } \mathcal{R}(\mathcal{S})}$$

- for symmetry $\pi \in SG$ and $\varepsilon = (\varepsilon_{ij})$ an **action π on a contraction matrix** $\varepsilon \mapsto \varepsilon^\pi$ is defined

$$(\varepsilon^\pi)_{ij} := \varepsilon_{\pi(i)\pi(j)}$$

- two solutions $\varepsilon_1, \varepsilon_2 \in \mathcal{R}(\mathcal{S})$ are **equivalent**, $\varepsilon_1 \sim \varepsilon_2$, if there exists normalization matrix α and $\pi \in SG$ that

$$\boxed{\varepsilon_1 = \alpha \bullet \varepsilon_2^\pi}$$

where $\bullet$ is componentwise matrix multiplication

- then holds

$$\boxed{\varepsilon_1 \sim \varepsilon_2 \;\Rightarrow\; \mathcal{L}^{\varepsilon_1} \simeq \mathcal{L}^{\varepsilon_2}}$$

- if contraction parameter ε_{ij} appears in $\mathcal{S}$, then $(i\ j)$ is **relevant** … $(i\ j) \in \mathcal{I}$

Process of solving

Theorem 1. Let $\mathcal{R}(\mathcal{S})$ be the set of solutions and $\mathcal{I}$ the set of relevant pairs of unordered indices of the contraction system $\mathcal{S}$ of a graded Lie algebra $\Gamma : \mathcal{L} = \bigoplus_{i \in G} \mathcal{L}_i$. For any $\mathcal{Q} \subset \mathcal{R}(\mathcal{S})$ and $\mathcal{P} = \{k_1,\ k_2, \ldots, k_m\} \subset \mathcal{I}$ we denote

$$\mathcal{R}_0 := \left\{ \varepsilon \in \mathcal{R}(\mathcal{S}) \middle| (\forall \varepsilon_1 \in \mathcal{Q})(\varepsilon \not\approx \varepsilon_1) \right\}$$
$$\mathcal{R}_1 := \left\{ \varepsilon \in \mathcal{R}_0 \,\middle|\, (\forall k \in \mathcal{P})(\varepsilon_k \neq 0) \right\}.$$

Then the solution $\varepsilon \in \mathcal{R}_0$ is non-equivalent to all solutions in $\mathcal{R}_1$ if and only if

$$\varepsilon_{\pi_1(k_1)} \varepsilon_{\pi_1(k_2)} \cdots \varepsilon_{\pi_1(k_m)} = 0$$
$$\vdots$$
$$\varepsilon_{\pi_n(k_1)} \varepsilon_{\pi_n(k_2)} \cdots \varepsilon_{\pi_n(k_m)} = 0$$

holds, where $\{\pi_1, \pi_2, \ldots, \pi_n\} = SG$ is the symmetry group of the grading Γ.

Overview of solutions

Grading	Equations	Solutions $\lvert\mathcal{R}(\mathcal{S})\rvert$						
	$\lvert\mathcal{S}\rvert$	All	0-p	1-p	2-p	3-p	4-p	5-p
Gell-Mann	44	89	79	8	2			
Cartan	31	48	27	9	6	4	1	1
Pauli	40	188	175	11	2			
Γ_d	37	978	480	288	152	49	8	1

Identification of results

- *Decomposition into a direct sum of indecomposable ideals*

$$\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2, \quad [\mathcal{L}_1, \mathcal{L}_2] = 0, \quad [\mathcal{L}_i, \mathcal{L}_i] \subseteq \mathcal{L}_i, \quad i = 1, 2.$$

- *Derived series* $D^0(\mathcal{L}) \supseteq D^1(\mathcal{L}) \supseteq \ldots \supseteq D^k(\mathcal{L}) \supseteq \ldots$

$$D^0(\mathcal{L}) = \mathcal{L}, \qquad D^{k+1}(\mathcal{L}) = [D^k(\mathcal{L}), D^k(\mathcal{L})], \qquad k = 0, 1, 2, \ldots$$

- *Lower central series* $\mathcal{L}^1 \supseteq \mathcal{L}^2 \supseteq \ldots \supseteq \mathcal{L}^k \supseteq \ldots$

$$\mathcal{L}^1 = \mathcal{L}, \qquad \mathcal{L}^{k+1} = [\mathcal{L}^k, \mathcal{L}], \qquad k = 0, 1, 2, \ldots$$

- *Upper central series* $C^0(\mathcal{L}) \subseteq C^1(\mathcal{L}) \subseteq \ldots \subseteq C^k(\mathcal{L}) \subseteq \ldots$

$$C^0(\mathcal{L}) = 0, \qquad C^{k+1}(\mathcal{L})/C^k(\mathcal{L}) = C(\mathcal{L}/C^k(\mathcal{L})), \qquad k = 0, 1, 2, \ldots$$

- *Number of formal invariants (Casimir operators)*

$$\tau(\mathcal{L}) = \dim(\mathcal{L}) - \sup_{(x_1,\ldots,x_n)} \operatorname{rank}(M_{\mathcal{L}}), \quad (M_{\mathcal{L}})_{ij} = \textstyle\sum_k c_{ij}^k x_k, \quad c_{ij}^k \text{ struc. const. of } \mathcal{L}$$

- *Levi decomposition, radical, nilradical*

- *Generalized derivations* $\mathcal{D}_{(1,1,1)}, \mathcal{D}_{(0,1,1)}, \mathcal{D}_{(1,1,0)}, \mathcal{D}_{(1,1,1)} \cap \mathcal{D}_{(0,1,1)}, \mathcal{D}_{(1,1,-1)}, \mathcal{D}_{(0,1,-1)}$

$$\mathcal{D}_{(\alpha,\beta,\gamma)} := \{A \in End(\mathcal{L}) \mid \alpha A[x, y] = \beta[Ax, y] + \gamma[x, Ay], \ \ \forall x, y \in \mathcal{L}\}$$

Overview of results

Grading	Algebras	Solvable	Nilpotent	Not solvable	Abelian
Pauli	148	21	125	1	1
Cartan	34	21	8	4	1
Gell-Mann	54	19	29	5	1

Pauli

Dimension of non–Abelian part	Solvable		Nilpotent		Total
	Indec.	Dec.	Indec.	Dec.	
3			1		1
4	1		1		2
5	1		4		5
6	1		9	1	11
7	4	1	28	1	34
8	11	2	77	3	93

Cartan

Dimension of non–Abelian part	Solvable		Nilpotent		Not solvable		Total
	Indec.	Dec.	Indec.	Dec.	Indec.	Dec.	
2	1						1
3	1		1		1		3
4		1					1
5	2	1	1				4
6	2	1	1	1	1		6
7	3	2	1		1		6
8	5	2	3		1		11

Gell–Mann

Dimension of non–Abelian part	Solvable		Nilpotent		Not solvable		Total
	Indec.	Dec.	Indec.	Dec.	Indec.	Dec.	
3	1		1				2
4							2
5	2		2				4
6	2	2	3	1	1	1	10
7	6		10				16
8	6		12		2		20

Contracted Lie algebras

Not solvable Lie algebras Gell-Mann graded $sl(3,\mathbb{C})$

Series	Algebra	Commutation relations	τ	T	Nilradical	$dim(\mathcal{D}(\alpha,\beta,\gamma))$	Name
(3)(3)(0)	$\mathcal{L}'_{2,1}$	$[e_1,e_2]=e_3,\ [e_1,e_3]=e_2,\ [e_2,e_3]=e_1$	1		$\emptyset$	$[3,0,1,0,0,1]$	$A_{3,8}$
(6)(6)(0)	$\mathcal{L}'_{4,2}$	$[e_1,e_5]=e_3,\ [e_1,e_6]=-e_2,\ [e_2,e_4]=-e_3,\ [e_2,e_6]=e_1,$ $[e_3,e_4]=e_2,\ [e_3,e_5]=-e_1,\ [e_4,e_5]=e_6,\ [e_4,e_6]=-e_5,\ [e_5,e_6]=e_4$	2		$3A_1$	$[7,0,2,0,0,2]$	$3A_1 \lhd A_{3,8}$
(8)(8)(0)	$\mathcal{L}_{1,2}$	$[e_1,e_6]=-e_4,\ [e_1,e_7]=-2e_2,\ [e_1,e_8]=e_3,\ [e_2,e_6]=-e_3,$ $[e_2,e_7]=-2e_1,\ [e_2,e_8]=e_4,\ [e_3,e_6]=-e_2,\ [e_3,e_7]=-e_4,$ $[e_3,e_8]=2e_5,\ [e_4,e_6]=-2e_1-2e_5,\ [e_4,e_7]=-e_3,\ [e_4,e_8]=-e_2,$ $[e_5,e_6]=-e_4,\ [e_5,e7]=e_2,\ [e_5,e_8]=-2e_3,\ [e_6,e_7]=-e_8,$ $[e_6,e_8]=-e_7,\ [e_7,e_8]=e_6$	2		$5A_1$	$[9,0,1,0,0,1]$	$5A_1 \lhd A_{3,8}$
(87)(87)(0)	$\mathcal{L}_{1,1}$	$[e_1,e_2]=-3e_3,\ [e_1,e_3]=-3e_2,\ [e_1,e_4]=-3e_5,\ [e_1,e_5]=-3e_4,$ $[e_2,e_6]=-e_5,\ [e_2,e_7]=e_4,\ [e_2,e_8]=e_3,\ [e_3,e_6]=-e_4,$ $[e_3,e_7]=e_5,\ [e_3,e_8]=e_2,\ [e_4,e_6]=e_3,\ [e_4,e_7]=e_2,$ $[e_4,e_8]=-e_5,\ [e_5,e_6]=e_2,\ [e_5,e_7]=e_3,\ [e_5,e_8]=-e_4,$ $[e_6,e_7]=-2e_8,\ [e_6,e_8]=2e_7,\ [e_7,e_8]=2e_6$	2		$4A_1$	$[9,0,1,0,0,1]$	$A_{5,7}(1,-1,-1) \lhd A_{3,8}$

Solvable Lie algebras Gell-Mann graded $sl(3,\mathbb{C})$

Series	Algebra	Commutation relations	τ	Nilradical	$dim(\mathcal{D}(\alpha,\beta,\gamma))$
(320)(32)(0)	$\mathcal{L}'_{11,2}$	$[e_1,e_2]=e_3,\ [e_1,e_3]=e_2$	1	$2A_1$	$[4,3,1,2,0,1]$
(530)(532)(12)	$\mathcal{L}'_{10,9}$	$[e_1,e_4]=e_3,\ [e_3,e_4]=e_1,\ [e_4,e_5]=e_2$	3	$4A_1$	$[8,9,3,5,2,6]$
(540)(54)(0)	$\mathcal{L}'_{9,8}(a)$	$[e_1,e_5]=ae_3,\ [e_2,e_5]=e_4,\ [e_3,e_5]=e_1,\ [e_4,e_5]=e_2$	3	$4A_1$	$[8,5,1,4,0,1]$
(640)(64)(0)	$\mathcal{L}'_{7,7}$	$[e_1,e_5]=e_2,\ [e_1,e_6]=e_4,\ [e_2,e_5]=e_1,$ $[e_2,e_6]=e_3,\ [e_3,e_5]=e_4,\ [e_4,e_5]=e_3$	2	$A_{5,1}$	$[9,6,2,4,0,2]$
(6510)(65)(1)	$\mathcal{L}'_{7,8}$	$[e_1,e_2]=e_5,\ [e_1,e_6]=e_4,\ [e_2,e_6]=e_3,$ $[e_3,e_4]=e_5,\ [e_3,e_6]=e_2,\ [e_4,e_6]=e_1$	2	$A_{5,4}$	$[10,6,2,1,1,7]$
(740)(742)(24)	$\mathcal{L}'_{9,6}$	$[e_1,e_5]=e_4,\ [e_4,e_5]=e_1,\ [e_5,e_6]=e_2,\ [e_5,e_7]=e_3$	5	$6A_1$	$[16,19,7,10,6,15]$
(750)(754)(1)	$\mathcal{L}'_{4,1}$	$[e_2,e_6]=e_4,\ [e_2,e_7]=e_3,\ [e_3,e_6]=e_5,\ [e_3,e_7]=e_2,\ [e_4,e_6]=e_2,$ $[e_4,e_7]=e_5,\ [e_5,e_6]=e_3,\ [e_5,e_7]=e_4,\ [e_6,e_7]=e_1$	3	$5A_1$	$[10,12,3,6,2,8]$
(750)(754)(1)	$\mathcal{L}'_{6,1}$	$[e_1,e_6]=e_4,\ [e_1,e_7]=e_2,\ [e_2,e_6]=e_5,\ [e_4,e_6]=e_1,$ $[e_4,e_7]=e_5,\ [e_5,e_6]=e_2,\ [e_6,e_7]=e_3$	3	$A_1\oplus A_{5,1}$	$[11,12,3,6,2,8]$
(750)(754)(12)	$\mathcal{L}'_{8,3}(a)$	$[e_1,e_6]=ae_4,\ [e_2,e_6]=e_5,\ [e_4,e_6]=e_1,\ [e_5,e_6]=e_2,$ $[e_6,e_7]=e_3$	5	$6A_1$	$[12,13,3,7,2,8]$
(7510)(75)(12)	$\mathcal{L}'_{6,2}$	$[e_1,e_2]=e_5,\ [e_1,e_7]=e_4,\ [e_2,e_7]=e_3,\ [e_3,e_4]=e_5,$ $[e_3,e_7]=e_2,\ [e_4,e_7]=e_1,\ [e_6,e_7]=e_5$	1	$A_1\oplus A_{5,4}$	$[12,12,3,3,2,8]$
(760)(76)(0)	$\mathcal{L}'_{7,2}(a,b)$	$[e_1,e_7]=ae_4,\ [e_2,e_7]=be_5,\ [e_3,e_7]=e_6,\ [e_4,e_7]=e_1,$ $[e_5,e_7]=e_2,\ [e_6,e_7]=e_3$	5	$6A_1$	$[12,7,1,6,0,1]$
(840)(842)(25)	$\mathcal{L}_{9,1}$	$[e_1,e_3]=2e_6,\ [e_1,e_4]=e_8,\ [e_1,e_5]=-e_7,\ [e_1,e_6]=2e_3,$ $[e_2,e_3]=e_6,\ [e_2,e_4]=-e_8,\ [e_2,e_5]=-2e_7,\ [e_2,e_6]=e_3$	4	$A_{5,1}$	$[16,21,9,12,8,17]$
(850)(854)(12)	$\mathcal{L}_{8,1}(a)$	$[e_1,e_3]=-2ae_6,\ [e_1,e_4]=-e_8,\ [e_1,e_5]=e_7,\ [e_1,e_6]=-2e_3,\ [e_1,e_7]=e_5,$ $[e_2,e_3]=ae_6,\ [e_2,e_4]=-e_8,\ [e_2,e_5]=-2e_7,\ [e_2,e_6]=e_3,\ [e_2,e_7]=-2e_5$	4	$6A_1$	$[13,14,4,8,3,9]$
(860)(86)(0)	$\mathcal{L}_{7,1}(a,b)$	$[e_1,e_3]=-2ae_6,\ [e_1,e_4]=-be_8,\ [e_1,e_5]=e_7,\ [e_1,e_6]=-2e_3,\ [e_1,e_7]=e_5$ $[e_1,e_8]=-e_4,\ [e_2,e_3]=ae_6,\ [e_2,e_4]=-be_8,\ [e_2,e_5]=-2e_7$ $[e_2,e_6]=e_3,\ [e_2,e_7]=-2e_5,\ [e_2,e_8]=-e_4$	4	$6A_1$	$[12,7,1,6,0,1]$
(8620)(86)(0)	$\mathcal{L}_{3,2}$	$[e_1,e_3]=2e_6,\ [e_1,e_4]=e_8,\ [e_1,e_5]=-e_7,\ [e_1,e_6]=2e_3,$ $[e_1,e_7]=-e_5,\ [e_1,e_8]=e_4,\ [e_2,e_3]=-e_6,\ [e_2,e_4]=e_8,$ $[e_2,e_5]=2e_7,\ [e_2,e_6]=-e_3,\ [e_2,e_7]=2e_5,\ [e_2,e_8]=e_4,$ $[e_3,e_5]=e_8,\ [e_3,e_7]=e_4,\ [e_5,e_6]=-e_4,\ [e_6,e_7]=e_8$	2	$A_{3,1}\oplus A_{3,1}$	$[10,2,1,2,0,1]$
	$\mathcal{L}_{3,1}$	$[e_1,e_3]=2e_6,\ [e_1,e_4]=e_8,\ [e_1,e_7]=-e_5,\ [e_2,e_3]=-e_6,$ $[e_2,e_4]=e_8,\ [e_2,e_7]=2e_5,\ [e_3,e_4]=e_7,\ [e_3,e_5]=e_8,$ $[e_3,e_6]=-2e_1,\ [e_3,e_7]=e_4,\ [e_3,e_8]=e_5,\ [e_4,e_6]=e_5,$ $[e_6,e_7]=e_8$	2		$[11,7,2,2,0,2]$
(8730)(87)(1)	$\mathcal{L}_{3,3}$	$[e_1,e_4]=-e_8,\ [e_2,e_4]=-e_8,\ [e_3,e_4]=-e_7,\ [e_3,e_5]=-e_8,$ $[e_3,e_6]=2e_1,\ [e_4,e_5]=-e_6,\ [e_4,e_6]=-e_5,\ [e_4,e_7]=e_3,$ $[e_4,e_8]=2e_1+2e_2,\ [e_5,e_7]=2e_2,\ [e_6,e_7]=-e_8$	2		$[12,10,2,3,1,9]$

Nilpotent Lie algebras Gell-Mann graded $sl(3,\mathbb{C})$

Series	Algebra	Commutation relations	τ	$dim(\mathcal{D}(\alpha,\beta,\gamma))$
(310)(310)(13)	$\mathcal{L}'_{12,3}$	$[e_2,e_3]=e_1$	1	$[6,6,3,5,3,4]$
(510)(510)(15)	$\mathcal{L}'_{11,4}$	$[e_2,e_5]=e_1,\ [e_3,e_4]=e_1$	1	$[15,15,5,14,10,11]$
(520)(520)(25)	$\mathcal{L}'_{11,3}$	$[e_3,e_4]=e_1,\ [e_4,e_5]=e_2$	3	$[13,13,7,9,7,11]$
(620)(620)(26)	$\mathcal{L}'_{10,7}$	$[e_3,e_5]=e_1,\ [e_4,e_5]=e_2,\ [e_4,e_6]=e_1$	2	$[17,18,10,14,10,14]$
(630)(630)(36)	$\mathcal{L}'_{10,16}$	$[e_4,e_5]=e_1,\ [e_4,e_6]=e_2,\ [e_5,e_6]=e_3$	4	$[18,18,10,9,10,19]$
(630)(6310)(136)	$\mathcal{L}'_{9,13}$	$[e_2,e_5]=e_1,\ [e_3,e_4]=e_1,\ [e_4,e_6]=e_2,\ [e_5,e_6]=e_3$	2	$[11,10,4,6,4,8]$
(710)(710)(17)	$\mathcal{L}'_{10,18}$	$[e_2,e_6]=e_1,\ [e_3,e_7]=e_1,\ [e_4,e_5]=e_1$	1	$[28,28,7,27,21,22]$
(720)(720)(27)	$\mathcal{L}'_{9,5}$	$[e_3,e_6]=e_1,\ [e_3,e_7]=e_2,\ [e_4,e_6]=e_1,$ $[e_4,e_7]=-2e_2,\ [e_5,e_6]=-e_2,\ [e_5,e_7]=e_1$	3	$[19,19,11,15,11,15]$
	$\mathcal{L}'_{10,5}$	$[e_3,e_5]=2e_2,\ [e_3,e_7]=-e_1,\ [e_4,e_5]=-e_2,\ [e_4,e_7]=2e_1,\ [e_5,e_6]=e_1$	3	$[21,22,11,18,14,18]$
(730)(730)(37)	$\mathcal{L}'_{9,16}$	$[e_4,e_6]=e_1,\ [e_4,e_7]=e_2,\ [e_5,e_6]=e_2,\ [e_5,e_7]=e_3$	3	$[19,24,13,15,13,22]$
	$\mathcal{L}'_{8,12}$	$[e_4,e_5]=e_1,\ [e_4,e_6]=e_2,\ [e_4,e_7]=e_3,\ [e_5,e_6]=e_3,\ [e_6,e_7]=e_1$	3	$[20,24,13,15,13,22]$
	$\mathcal{L}'_{9,12}$	$[e_4,e_5]=e_3,\ [e_5,e_6]=e_1,\ [e_5,e_7]=e_2,\ [e_6,e_7]=e_3$	3	$[22,24,13,15,13,22]$
	$\mathcal{L}'_{10,6}$	$[e_4,e_5]=e_1,\ [e_5,e_6]=e_2,\ [e_5,e_7]=e_3$	5	$[25,25,13,16,13,22]$
(730)(7310)(147)	$\mathcal{L}'_{8,8}$	$[e_2,e_6]=e_1,\ [e_3,e_5]=e_1,\ [e_4,e_5]=e_2,\ [e_4,e_6]=e_3,\ [e_4,e_7]=e_1$	1	$[15,16,7,10,7,11]$
(740)(7410)(147)	$\mathcal{L}'_{7,6}(a)$	$[e_1,e_7]=ae_2,\ [e_3,e_6]=(a+1)e_2,\ [e_4,e_5]=e_2,$ $[e_5,e_6]=e_1,\ [e_5,e_7]=e_3,\ [e_6,e_7]=e_4\quad a\neq -1$	1	$[15,13,7,9,6,11]$
(740)(7410)(247)	$\mathcal{L}'_{8,7}$	$[e_3,e_6]=e_1,\ [e_4,e_5]=e_1,\ [e_5,e_6]=e_2,\ [e_5,e_7]=e_3,\ [e_6,e_7]=e_4$	3	$[15,17,8,9,7,16]$
(820)(820)(28)	$\mathcal{L}_{10,23}$	$[e_3,e_6]=e_1,\ [e_4,e_8]=e_1+e_2,\ [e_5,e_7]=e_2$	2	$[22,25,13,21,15,19]$
(830)(830)(38)	$\mathcal{L}_{7,3}$	$[e_1,e_3]=2e_6,\ [e_1,e_4]=e_8,\ [e_1,e_5]=-e_7,\ [e_2,e_3]=-e_6,$ $[e_2,e_4]=e_8,\ [e_2,e_5]=2e_7,\ [e_3,e_4]=e_7,\ [e_3,e_5]=e_8,\ [e_4,e_5]=e_6$	4	$[19,24,16,15,16,25]$
	$\mathcal{L}_{8,2}$	$[e_1,e_3]=2e_6,\ [e_1,e_4]=e_8,\ [e_1,e_5]=-e_7,\ [e_2,e_3]=-e_6,\ [e_2,e_4]=e_8$ $[e_2,e_5]=2e_7,\ [e_3,e_4]=e_7,\ [e_3,e_5]=e_8$	4	$[20,24,16,15,16,25]$
	$\mathcal{L}_{9,3}$	$[e_1,e_3]=2e_6,\ [e_1,e_4]=e_8,\ [e_1,e_5]=-e_7,\ [e_2,e_3]=-e_6$ $[e_2,e_4]=e_8,\ [e_2,e_5]=2e_7,\ [e_3,e_4]=e_7$	4	$[21,25,16,16,16,25]$
	$\mathcal{L}_{10,1}$	$[e_1,e_3]=2e_6,\ [e_1,e_4]=e_8,[e_1,e_5]=-e_7,\ [e_2,e_3]=-e_6,$ $[e_2,e_4]=e_8,\ [e_2,e_5]=2e_7$	4	$[22,28,16,19,16,25]$
	$\mathcal{L}_{8,6}$	$[e_1,e_3]=2e_6,\ [e_1,e_4]=e_8,\ [e_2,e_3]=-e_6,\ [e_2,e_4]=e_8,$ $[e_3,e_4]=e_7,\ [e_3,e_5]=e_8,\ [e_4,e_5]=e_6$	4	$[26,28,16,19,16,25]$
	$\mathcal{L}_{9,4}$	$[e_1,e_3]=2e_6,\ [e_1,e_4]=e_8,\ [e_2,e_3]=-e_6,\ [e_2,e_4]=e_8,$ $[e_3,e_4]=e_7,\ [e_3,e_5]=e_8$	4	$[27,30,17,21,17,26]$
(840)(840)(48)	$\mathcal{L}_{7,9}$	$[e_3,e_5]=e_8,\ [e_3,e_6]=e_1,\ [e_3,e_7]=e_4,\ [e_5,e_6]=e_4,\ [e_5,e_7]=e_2,\ [e_6,e_7]=e_8$	4	$[24,33,17,17,17,33]$
	$\mathcal{L}_{8,9}$	$[e_3,e_4]=e_7,\ [e_3,e_6]=e_1,\ [e_3,e_8]=e_5,\ [e_4,e_6]=e_5,\ [e_4,e_8]=e_1+e_2$	4	$[25,33,17,17,17,33]$
	$\mathcal{L}_{9,15}$	$[e_3,e_4]=e_7,\ [e_3,e_6]=e_1,\ [e_3,e_8]=e_5,\ [e_4,e_8]=e_1+e_2$	4	$[27,33,17,17,17,33]$
(840)(8410)(148)	$\mathcal{L}_{7,4}$	$[e_1,e_3]=2e_6,\ [e_1,e_4]=e_8,\ [e_1,e_7]=-e_5,\ [e_2,e_3]=-e_6,\ [e_2,e_4]=e_8,$ $[e_2,e_7]=2e_5,\ [e_3,e_4]=e_7,\ [e_3,e_8]=e_5,\ [e_4,e_6]=e_5$	2	$[18,16,6,10,7,12]$
(850)(8520)(258)	$\mathcal{L}_{7,5}$	$[e_3,e_4]=e_7,\ [e_3,e_5]=e_8,\ [e_3,e_6]=e_1,\ [e_4,e_5]=e_6,$ $[e_4,e_8]=e_1+e_2,\ [e_5,e_7]=e_2$	2	$[18,19,7,9,6,17]$

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