CZECH TECHNICAL UNIVERSITY IN PRAGUE
Faculty of Nuclear Sciences and Physical Engineering

Twisted Cocycles of Lie Algebras and Corresponding Invariant Functions

Stará Lesná, September 2007

J. Hrivnák, P. Novotný

Faculty of Nuclear Sciences and Physical Engineering
Czech Technical University in Prague
Břehová 7, CZ-115 19 Prague, Czech Republic

- q–dimensional twisted cocycles of the adjoint representation

- q–dimensional twisted cocycles of the adjoint representation
- $q = 1$... generalized derivations
  - classification of (α, β, γ)–derivations
  - three–dimensional Lie algebras

- q–dimensional twisted cocycles of the adjoint representation

- $q = 1$... generalized derivations

  - classification of (α, β, γ)–derivations
  - three–dimensional Lie algebras

- two–dimensional twisted cocycles of the adjoint representation

  - classification
  - four–dimensional Lie algebras

- q–dimensional twisted cocycles of the adjoint representation

- $q = 1$... generalized derivations
  - classification of (α, β, γ)–derivations
  - three–dimensional Lie algebras

- two–dimensional twisted cocycles of the adjoint representation
  - classification
  - four–dimensional Lie algebras

- application to contractions
  - one–parametric continuous contractions
  - graded contractions

Notation

- $\mathcal{L}$ . . . finite–dimensional complex Lie algebra

Notation

- $\mathcal{L}\ldots$ finite–dimensional complex Lie algebra
- $\mathrm{End}(\mathcal{L})\ldots$ associative algebra of all endomorphisms of the vector space $\mathcal{L}$

Notation

- $\mathcal{L} \ldots$ finite–dimensional complex Lie algebra
- $\mathrm{End}(\mathcal{L}) \ldots$ associative algebra of all endomorphisms of the vector space $\mathcal{L}$
- $\mathrm{gl}(\mathcal{L}) \ldots [A, B] = AB - BA \qquad A, B \in \mathrm{End}(\mathcal{L})$

Notation

- $\mathcal{L}\ldots$ finite–dimensional complex Lie algebra

- $\mathrm{End}(\mathcal{L})\ldots$ associative algebra of all endomorphisms of the vector space $\mathcal{L}$

- $\mathrm{gl}(\mathcal{L})\ldots[A,B]=AB-BA \qquad A,B\in\mathrm{End}(\mathcal{L})$

- $\mathrm{jor}(\mathcal{L})\ldots A\circ B=\frac{1}{2}(AB+BA) \qquad A,B\in\mathrm{End}(\mathcal{L})$

Notation

- $\mathcal{L} \ldots$ finite–dimensional complex Lie algebra
- $\mathrm{End}(\mathcal{L}) \ldots$ associative algebra of all endomorphisms of the vector space $\mathcal{L}$
- $\mathrm{gl}(\mathcal{L}) \ldots [A, B] = AB - BA \qquad A, B \in \mathrm{End}(\mathcal{L})$
- $\mathrm{jor}(\mathcal{L}) \ldots A \circ B = \frac{1}{2}(AB + BA) \qquad A, B \in \mathrm{End}(\mathcal{L})$
- $\mathrm{der}(\mathcal{L}) \ldots$ Lie algebra of derivations of $\mathcal{L}$

Notation

- $\mathcal{L} \ldots$ finite–dimensional complex Lie algebra
- $\mathrm{End}(\mathcal{L}) \ldots$ associative algebra of all endomorphisms of the vector space $\mathcal{L}$
- $\mathrm{gl}(\mathcal{L}) \ldots [A, B] = AB - BA \qquad A, B \in \mathrm{End}(\mathcal{L})$
- $\mathrm{jor}(\mathcal{L}) \ldots A \circ B = \frac{1}{2}(AB + BA) \qquad A, B \in \mathrm{End}(\mathcal{L})$
- $\mathrm{der}(\mathcal{L}) \ldots$ Lie algebra of derivations of $\mathcal{L}$
- $C(\mathcal{L}) \ldots$ center of Lie algebra $\mathcal{L}$

Notation

- $\mathcal{L} \ldots$ finite–dimensional complex Lie algebra
- $\mathrm{End}(\mathcal{L}) \ldots$ associative algebra of all endomorphisms of the vector space $\mathcal{L}$
- $\mathrm{gl}(\mathcal{L}) \ldots [A, B] = AB - BA \qquad A, B \in \mathrm{End}(\mathcal{L})$
- $\mathrm{jor}(\mathcal{L}) \ldots A \circ B = \frac{1}{2}(AB + BA) \qquad A, B \in \mathrm{End}(\mathcal{L})$
- $\mathrm{der}(\mathcal{L}) \ldots$ Lie algebra of derivations of $\mathcal{L}$
- $C(\mathcal{L}) \ldots$ center of Lie algebra $\mathcal{L}$
- $C^q(\mathcal{L}, \mathcal{L}) \ldots$ space of q–dimensional $\mathcal{L}$–cochains, i. e. $q-$linear maps

$$c : \underbrace{\mathcal{L} \times \mathcal{L} \times \cdots \times \mathcal{L}}_{q-times} \to \mathcal{L}$$

Notation

- $\mathcal{L}\ldots$ finite–dimensional complex Lie algebra

- $\mathrm{End}(\mathcal{L})\ldots$ associative algebra of all endomorphisms of the vector space $\mathcal{L}$

- $\mathrm{gl}(\mathcal{L})\ldots [A,B]=AB-BA \qquad A,B\in\mathrm{End}(\mathcal{L})$

- $\mathrm{jor}(\mathcal{L})\ldots A\circ B=\frac{1}{2}(AB+BA) \qquad A,B\in\mathrm{End}(\mathcal{L})$

- $\mathrm{der}(\mathcal{L})\ldots$ Lie algebra of derivations of $\mathcal{L}$

- $C(\mathcal{L})\ldots$ center of Lie algebra $\mathcal{L}$

- $C^q(\mathcal{L},\mathcal{L})\ldots$ space of q–dimensional $\mathcal{L}$–cochains, i. e. $q-$linear maps

$$c:\underbrace{\mathcal{L}\times\mathcal{L}\times\cdots\times\mathcal{L}}_{q-times}\to\mathcal{L}$$

$$c(x_1,\ldots,x_i,\ldots,x_j,\ldots,x_q)+c(x_1,\ldots,x_j,\ldots,x_i,\ldots,x_q)=0$$

Twisted Cocycles of Lie algebras

- $Z^q(\mathcal{L}, \mathrm{ad}) \ldots$ q–dimensional cocycles of the adjoint representation

Twisted Cocycles of Lie algebras

- $Z^q(\mathcal{L}, \mathrm{ad}) \ldots$ q–dimensional cocycles of the adjoint representation: $c \in C^q(\mathcal{L}, \mathcal{L})$

$$0 = \sum_{i=1}^{q+1} (-1)^{i+1} [x_i, c(x_1, \ldots, \hat{x}_i, \ldots, x_{q+1})] +$$

$$+ \sum_{\substack{i,j=1 \\ i<j}}^{q+1} (-1)^{i+j} c([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{q+1})$$

Twisted Cocycles of Lie algebras

- $Z^q(\mathcal{L}, \mathrm{ad}) \ldots$ q–dimensional cocycles of the adjoint representation: $c \in C^q(\mathcal{L}, \mathcal{L})$

$$0 = \sum_{i=1}^{q+1} (-1)^{i+1} [x_i, c(x_1, \ldots, \hat{x}_i, \ldots, x_{q+1})] + \\ + \sum_{\substack{i,j=1 \\ i<j}}^{q+1} (-1)^{i+j} c([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{q+1})$$

- generalization: $\kappa \ldots$ $(q+1) \times (q+1)$ complex symmetric matrix
  $Z^q(\mathcal{L}, \mathrm{ad}, \kappa) \ldots q$–dimensional κ–cocycles of the adjoint representation: $c \in C^q(\mathcal{L}, \mathcal{L})$

Twisted Cocycles of Lie algebras

- $Z^q(\mathcal{L}, \mathrm{ad}) \ldots$ q–dimensional cocycles of the adjoint representation: $c \in C^q(\mathcal{L}, \mathcal{L})$

$$0 = \sum_{i=1}^{q+1} (-1)^{i+1} [x_i, c(x_1, \ldots, \hat{x}_i, \ldots, x_{q+1})] + \\ + \sum_{\substack{i,j=1 \\ i<j}}^{q+1} (-1)^{i+j} c([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{q+1})$$

- generalization: $\kappa \ldots (q+1) \times (q+1)$ complex symmetric matrix
  $Z^q(\mathcal{L}, \mathrm{ad}, \kappa) \ldots q$–dimensional κ–cocycles of the adjoint representation: $c \in C^q(\mathcal{L}, \mathcal{L})$

$$0 = \sum_{i=1}^{q+1} (-1)^{i+1} \kappa_{ii} [x_i, c(x_1, \ldots, \hat{x}_i, \ldots, x_{q+1})] + \\ + \sum_{\substack{i,j=1 \\ i<j}}^{q+1} (-1)^{i+j} \kappa_{ij} c([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{q+1})$$

- **Proposition.** Let $g : \mathcal{L} \to \widetilde{\mathcal{L}}$ be an isomorphism of Lie algebras $\mathcal{L}$ and $\widetilde{\mathcal{L}}$. Then the mapping $\varrho : C^q(\mathcal{L},\mathcal{L}) \to C^q(\widetilde{\mathcal{L}},\widetilde{\mathcal{L}})$, $q \geq 1$ defined for all $c \in C^q(\mathcal{L},\mathcal{L})$ and all $x_1,\dots,x_q \in \widetilde{\mathcal{L}}$ by

$$(\varrho c)(x_1,\dots,x_q) = gc(g^{-1}x_1,\dots,g^{-1}x_q)$$

is an isomorphism of vector spaces $C^q(\mathcal{L},\mathcal{L})$ and $C^q(\widetilde{\mathcal{L}},\widetilde{\mathcal{L}})$. For any complex symmetric $(q+1)$–square matrix κ

$$\boxed{\varrho(Z^q(\mathcal{L},\mathrm{ad}_{\mathcal{L}},\kappa)) = Z^q(\widetilde{\mathcal{L}},\mathrm{ad}_{\widetilde{\mathcal{L}}},\kappa)}$$

holds.

- **Proposition.** Let $g : \mathcal{L} \to \widetilde{\mathcal{L}}$ be an isomorphism of Lie algebras $\mathcal{L}$ and $\widetilde{\mathcal{L}}$. Then the mapping $\varrho : C^q(\mathcal{L},\mathcal{L}) \to C^q(\widetilde{\mathcal{L}},\widetilde{\mathcal{L}})$, $q \geq 1$ defined for all $c \in C^q(\mathcal{L},\mathcal{L})$ and all $x_1,\ldots,x_q \in \widetilde{\mathcal{L}}$ by

$$(\varrho c)(x_1,\ldots,x_q) = gc(g^{-1}x_1,\ldots,g^{-1}x_q)$$

is an isomorphism of vector spaces $C^q(\mathcal{L},\mathcal{L})$ and $C^q(\widetilde{\mathcal{L}},\widetilde{\mathcal{L}})$. For any complex symmetric $(q+1)$–square matrix κ

$$\boxed{\varrho(Z^q(\mathcal{L},\mathrm{ad}_{\mathcal{L}},\kappa)) = Z^q(\widetilde{\mathcal{L}},\mathrm{ad}_{\widetilde{\mathcal{L}}},\kappa)}$$

holds.

$\Rightarrow$ dimension of $Z^q(\mathcal{L},\mathrm{ad},\kappa)$ is an invariant of Lie algebra $\mathcal{L}$

(α, β, γ)–derivations

- $q = 1$

(α, β, γ)–derivations

- $q = 1$
- $Z^1(\mathcal{L}, \mathrm{ad}) = \mathrm{der}(\mathcal{L})$

(α, β, γ)–derivations

- $q = 1$
- $Z^1(\mathcal{L}, \mathrm{ad}) = \mathrm{der}(\mathcal{L})$
- $Z^1\left(\mathcal{L}, \mathrm{ad}, \left(\begin{smallmatrix} \beta & \alpha \\ \alpha & \gamma \end{smallmatrix}\right)\right) = \mathrm{der}_{(\alpha,\beta,\gamma)}\, \mathcal{L}$

(α, β, γ)–derivations

- $q = 1$
- $Z^1(\mathcal{L}, \mathrm{ad}) = \mathrm{der}(\mathcal{L})$
- $Z^1\left(\mathcal{L}, \mathrm{ad}, \left(\begin{smallmatrix} \beta & \alpha \\ \alpha & \gamma \end{smallmatrix}\right)\right) = \mathrm{der}_{(\alpha,\beta,\gamma)}\ \mathcal{L}$

$$\mathrm{der}_{(\alpha,\beta,\gamma)}\ \mathcal{L} = \{A \in \mathrm{End}(\mathcal{L}) \mid \alpha A[x,y] = \beta[Ax,y] + \gamma[x,Ay], \quad \forall x,y \in \mathcal{L}\}$$

(α, β, γ)–derivations

- $q = 1$
- $Z^1(\mathcal{L}, \mathrm{ad}) = \mathrm{der}(\mathcal{L})$
- $Z^1\left(\mathcal{L}, \mathrm{ad}, \left(\begin{smallmatrix} \beta & \alpha \\ \alpha & \gamma \end{smallmatrix}\right)\right) = \mathrm{der}_{(\alpha,\beta,\gamma)}\, \mathcal{L}$

$$\mathrm{der}_{(\alpha,\beta,\gamma)}\, \mathcal{L} = \{A \in \mathrm{End}(\mathcal{L}) \mid \alpha A[x,y] = \beta[Ax,y] + \gamma[x,Ay], \ \ \forall x, y \in \mathcal{L}\}$$

- **Theorem.** For any $\alpha, \beta, \gamma \in \mathbb{C}$ there exists $\delta \in \mathbb{C}$ such that the subspace $\mathrm{der}_{(\alpha,\beta,\gamma)}\, \mathcal{L} \subset \mathrm{End}(\mathcal{L})$ is equal to some of the four following subspaces:
  1. $\mathrm{der}_{(\delta,0,0)}\, \mathcal{L}$
  2. $\mathrm{der}_{(\delta,1,-1)}\, \mathcal{L}$
  3. $\mathrm{der}_{(\delta,1,0)}\, \mathcal{L}$
  4. $\mathrm{der}_{(\delta,1,1)}\, \mathcal{L}$.

Theorem. Suppose that $\alpha, \beta, \gamma \in \mathbb{C}$ are not all zero. Then the space $\operatorname{der}_{(\alpha,\beta,\gamma)} \mathcal{L}$ is equal to some of the following:

1. Lie algebra of derivations $\operatorname{der}_{(1,1,1)} \mathcal{L} \subset \operatorname{gl}(\mathcal{L})$,

2. Lie algebra $\operatorname{der}_{(0,1,1)} \mathcal{L} \subset \operatorname{gl}(\mathcal{L})$,

Theorem. Suppose that $\alpha, \beta, \gamma \in \mathbb{C}$ are not all zero. Then the space $\operatorname{der}_{(\alpha,\beta,\gamma)} \mathcal{L}$ is equal to some of the following:

1. Lie algebra of derivations $\operatorname{der}_{(1,1,1)} \mathcal{L} \subset \mathrm{gl}(\mathcal{L})$,
2. Lie algebra $\operatorname{der}_{(0,1,1)} \mathcal{L} \subset \mathrm{gl}(\mathcal{L})$,
3. associative algebra $\operatorname{der}_{(1,1,0)} \mathcal{L} \subset \operatorname{End}(\mathcal{L})$ (centralizer of the adjoint representation)
4. associative algebra $\operatorname{der}_{(1,0,0)} \mathcal{L} \subset \operatorname{End}(\mathcal{L})$ of dimension
$$\dim \operatorname{der}_{(1,0,0)} \mathcal{L} = \operatorname{codim}[\mathcal{L}, \mathcal{L}] \dim \mathcal{L},$$
5. associative algebra $\operatorname{der}_{(0,1,0)} \mathcal{L} \subset \operatorname{End}(\mathcal{L})$ of dimension
$$\dim \operatorname{der}_{(0,1,0)} \mathcal{L} = \dim \mathcal{L} \, \dim C(\mathcal{L}),$$

Theorem. Suppose that $\alpha, \beta, \gamma \in \mathbb{C}$ are not all zero. Then the space $\operatorname{der}_{(\alpha,\beta,\gamma)} \mathcal{L}$ is equal to some of the following:

1. Lie algebra of derivations $\operatorname{der}_{(1,1,1)} \mathcal{L} \subset \operatorname{gl}(\mathcal{L})$,
2. Lie algebra $\operatorname{der}_{(0,1,1)} \mathcal{L} \subset \operatorname{gl}(\mathcal{L})$,
3. associative algebra $\operatorname{der}_{(1,1,0)} \mathcal{L} \subset \operatorname{End}(\mathcal{L})$ (centralizer of the adjoint representation)
4. associative algebra $\operatorname{der}_{(1,0,0)} \mathcal{L} \subset \operatorname{End}(\mathcal{L})$ of dimension
$$\dim \operatorname{der}_{(1,0,0)} \mathcal{L} = \operatorname{codim}[\mathcal{L}, \mathcal{L}] \dim \mathcal{L},$$
5. associative algebra $\operatorname{der}_{(0,1,0)} \mathcal{L} \subset \operatorname{End}(\mathcal{L})$ of dimension
$$\dim \operatorname{der}_{(0,1,0)} \mathcal{L} = \dim \mathcal{L} \, \dim C(\mathcal{L}),$$
6. Jordan algebra $\operatorname{der}_{(1,1,-1)} \mathcal{L} \subset \operatorname{jor}(\mathcal{L})$,
7. Jordan algebra $\operatorname{der}_{(0,1,-1)} \mathcal{L} \subset \operatorname{jor}(\mathcal{L})$ (quasicentroid),

Theorem. Suppose that $\alpha, \beta, \gamma \in \mathbb{C}$ are not all zero. Then the space $\operatorname{der}_{(\alpha,\beta,\gamma)} \mathcal{L}$ is equal to some of the following:

1. Lie algebra of derivations $\operatorname{der}_{(1,1,1)} \mathcal{L} \subset \operatorname{gl}(\mathcal{L})$,
2. Lie algebra $\operatorname{der}_{(0,1,1)} \mathcal{L} \subset \operatorname{gl}(\mathcal{L})$,
3. associative algebra $\operatorname{der}_{(1,1,0)} \mathcal{L} \subset \operatorname{End}(\mathcal{L})$ (centralizer of the adjoint representation)
4. associative algebra $\operatorname{der}_{(1,0,0)} \mathcal{L} \subset \operatorname{End}(\mathcal{L})$ of dimension
$$\dim \operatorname{der}_{(1,0,0)} \mathcal{L} = \operatorname{codim}[\mathcal{L}, \mathcal{L}] \dim \mathcal{L},$$
5. associative algebra $\operatorname{der}_{(0,1,0)} \mathcal{L} \subset \operatorname{End}(\mathcal{L})$ of dimension
$$\dim \operatorname{der}_{(0,1,0)} \mathcal{L} = \dim \mathcal{L} \, \dim C(\mathcal{L}),$$
6. Jordan algebra $\operatorname{der}_{(1,1,-1)} \mathcal{L} \subset \operatorname{jor}(\mathcal{L})$,
7. Jordan algebra $\operatorname{der}_{(0,1,-1)} \mathcal{L} \subset \operatorname{jor}(\mathcal{L})$ (quasicentroid),
8. subspace $\operatorname{der}_{(\delta,1,0)} \mathcal{L}$ for some $\delta \in \mathbb{C}$,
9. subspace $\operatorname{der}_{(\delta,1,1)} \mathcal{L}$ for some $\delta \in \mathbb{C}$.

Intersections of the spaces $\mathrm{der}_{(\alpha,\beta,\gamma)}\, \mathcal{L}$

Intersections of the spaces $\operatorname{der}_{(\alpha,\beta,\gamma)} \mathcal{L}$

- **Theorem.** Let $\mathcal{L}$ be a complex Lie algebra. Suppose $\alpha, \beta, \gamma \in \mathbb{C}$ are not all zero and $\alpha', \beta', \gamma' \in \mathbb{C}$. Then the intersection $\operatorname{der}_{(\alpha,\beta,\gamma)} \mathcal{L} \cap \operatorname{der}_{(\alpha',\beta',\gamma')} \mathcal{L}$ is equal to some of the previous cases 1. – 9. or to some of the following:

  1. associative algebra $\operatorname{der}_{(1,0,0)} \mathcal{L} \cap \operatorname{der}_{(0,1,0)} \mathcal{L} \subset \operatorname{End} \mathcal{L}$ of dimension
  $$\dim(\operatorname{der}_{(1,0,0)} \mathcal{L} \cap \operatorname{der}_{(0,1,0)} \mathcal{L}) = \operatorname{codim}[\mathcal{L}, \mathcal{L}] \dim C(\mathcal{L}).$$
  2. Lie algebra $\operatorname{der}_{(1,1,1)} \mathcal{L} \cap \operatorname{der}_{(0,1,1)} \mathcal{L} \subset \operatorname{gl}(\mathcal{L})$.

Invariant functions

- $\psi\,\mathcal{L} : \mathbb{C} \to \{0, 1, 2, \ldots, (\dim \mathcal{L})^2\}$

$$\psi\,\mathcal{L}(\alpha) = \dim \operatorname{der}_{(\alpha,1,1)} \mathcal{L}$$

Invariant functions

- $\psi\,\mathcal{L} : \mathbb{C} \to \{0, 1, 2, \ldots, (\dim \mathcal{L})^2\}$

$$\psi\,\mathcal{L}(\alpha) = \dim \mathrm{der}_{(\alpha,1,1)}\ \mathcal{L}$$

- $\psi^0\,\mathcal{L} : \mathbb{C} \to \{0, 1, 2, \ldots, (\dim \mathcal{L})^2\}$

$$\psi^0\,\mathcal{L}(\alpha) = \dim \mathrm{der}_{(\alpha,1,0)}\ \mathcal{L}$$

Invariant functions

- $\psi\,\mathcal{L} : \mathbb{C} \to \{0, 1, 2, \ldots, (\dim \mathcal{L})^2\}$

$$\psi\,\mathcal{L}(\alpha) = \dim \operatorname{der}_{(\alpha,1,1)}\,\mathcal{L}$$

- $\psi^0\,\mathcal{L} : \mathbb{C} \to \{0, 1, 2, \ldots, (\dim \mathcal{L})^2\}$

$$\psi^0\,\mathcal{L}(\alpha) = \dim \operatorname{der}_{(\alpha,1,0)}\,\mathcal{L}$$

$$\mathcal{L} \cong \widetilde{\mathcal{L}} \;\Rightarrow\; \psi\,\mathcal{L} = \psi\,\widetilde{\mathcal{L}} \text{ and } \psi^0\,\mathcal{L} = \psi^0\,\widetilde{\mathcal{L}}$$

Example

- $\mathrm{g}_{3,4}(a)$: $\quad [e_1, e_3] = e_1,\ [e_2, e_3] = ae_2,\ a \neq 0, \pm 1$

Example

- $\mathfrak{g}_{3,4}(a):\quad [e_1,e_3]=e_1,\ [e_2,e_3]=ae_2,\ a\neq 0,\pm 1$

  $\cdot\ \mathrm{der}_{(1,1,1)} = \mathrm{span}_{\mathbb{C}}\left\{\begin{pmatrix} 0&0&1\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix} 0&0&0\\0&0&1\\0&0&0\end{pmatrix},\begin{pmatrix} 1&0&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix} 0&0&0\\0&1&0\\0&0&0\end{pmatrix}\right\}\cong \mathfrak{g}_{2,1}\oplus\mathfrak{g}_{2,1}$

  $\cdot\ \mathrm{der}_{(0,1,1)} = \mathrm{span}_{\mathbb{C}}\left\{\begin{pmatrix} 0&0&1\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix} 0&0&0\\0&0&1\\0&0&0\end{pmatrix},\begin{pmatrix} 1&0&0\\0&1&0\\0&0&-1\end{pmatrix}\right\}\cong \mathfrak{g}_{3,3}$

  $\cdot\ \mathrm{der}_{(1,1,0)} = \mathrm{span}_{\mathbb{C}}\left\{\begin{pmatrix} 1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right\}\cong \mathfrak{g}_{1}$

  $\cdot\ \mathrm{der}_{(1,0,0)} = \mathrm{span}_{\mathbb{C}}\left\{\begin{pmatrix} 0&0&1\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix} 0&0&0\\0&0&1\\0&0&0\end{pmatrix},\begin{pmatrix} 0&0&0\\0&0&0\\0&0&1\end{pmatrix}\right\}\cong \mathfrak{g}_{3,3}$

  $\cdot\ \mathrm{der}_{(0,1,0)} = \{0\}$

  $\cdot\ \mathrm{der}_{(1,1,1)}\cap\mathrm{der}_{(0,1,1)} = \mathrm{span}_{\mathbb{C}}\left\{\begin{pmatrix} 0&0&1\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix} 0&0&0\\0&0&1\\0&0&0\end{pmatrix}\right\}\cong 2\,\mathfrak{g}_{1}$

  $\cdot\ \mathrm{der}_{(1,0,0)}\cap\mathrm{der}_{(0,1,0)} = \{0\}$

  $\cdot\ \mathrm{der}_{(1,1,-1)} = \{0\}$

$$\cdot\ \mathrm{der}_{(0,1,-1)} = \mathrm{span}_{\mathbb{C}}\left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}$$

$$\cdot\ \mathrm{der}_{(\delta,1,0)} = \{0\}_{\delta\neq 1}$$

$$\cdot\ \mathrm{der}_{(\delta,1,1)} = \mathrm{span}_{\mathbb{C}}\left\{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1+\delta \end{pmatrix}\right\}_{\delta\neq 1,a,1/a}$$

$$\cdot\ \mathrm{der}_{(a,1,1)} = \mathrm{span}_{\mathbb{C}}\left\{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1+a \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right\}$$

$$\cdot\ \mathrm{der}_{(\frac{1}{a},1,1)} = \mathrm{span}_{\mathbb{C}}\left\{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1+\frac{1}{a} \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right\}$$

$$\cdot\ \mathrm{der}_{(0,1,-1)} = \mathrm{span}_{\mathbb{C}}\left\{\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right\}$$

$$\cdot\ \mathrm{der}_{(\delta,1,0)} = \{0\}_{\delta\neq 1}$$

$$\cdot\ \mathrm{der}_{(\delta,1,1)} = \mathrm{span}_{\mathbb{C}}\left\{\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix},\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1+\delta\end{pmatrix}\right\}_{\delta\neq 1,a,1/a}$$

$$\cdot\ \mathrm{der}_{(a,1,1)} = \mathrm{span}_{\mathbb{C}}\left\{\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix},\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1+a\end{pmatrix},\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}\right\}$$

$$\cdot\ \mathrm{der}_{(\frac{1}{a},1,1)} = \mathrm{span}_{\mathbb{C}}\left\{\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix},\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1+\frac{1}{a}\end{pmatrix},\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}\right\}$$

α	1	a	$\frac{1}{a}$	
$[\psi\, \mathrm{g}_{3,4}(a)](\alpha)$	4	4	4	3

α	1	
$[\psi^0\, \mathrm{g}_{3,4}(a)](\alpha)$	1	0

$$\cdot\ \mathrm{der}_{(0,1,-1)} = \mathrm{span}_{\mathbb{C}}\left\{\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right\}$$

$$\cdot\ \mathrm{der}_{(\delta,1,0)} = \{0\}_{\delta\neq 1}$$

$$\cdot\ \mathrm{der}_{(\delta,1,1)} = \mathrm{span}_{\mathbb{C}}\left\{\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix},\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1+\delta\end{pmatrix}\right\}_{\delta\neq 1,a,1/a}$$

$$\cdot\ \mathrm{der}_{(a,1,1)} = \mathrm{span}_{\mathbb{C}}\left\{\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix},\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1+a\end{pmatrix},\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}\right\}$$

$$\cdot\ \mathrm{der}_{(\frac{1}{a},1,1)} = \mathrm{span}_{\mathbb{C}}\left\{\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix},\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1+\frac{1}{a}\end{pmatrix},\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}\right\}$$

α	1	a	$\frac{1}{a}$	
$[\psi\,\mathfrak{g}_{3,4}(a)](\alpha)$	4	4	4	3

α	1	
$[\psi^0\,\mathfrak{g}_{3,4}(a)](\alpha)$	1	0

$$\mathfrak{g}_{3,4}(a) \cong \mathfrak{g}_{3,4}(b) \Leftrightarrow \psi\,\mathfrak{g}_{3,4}(a) = \psi\,\mathfrak{g}_{3,4}(b).$$

<table>
<tr><th>$\mathcal{L}$</th><th>Commutators</th><th></th><th>Invariant function</th></tr>
<tr><td>$\mathrm{g}_{2,1} \oplus \mathrm{g}_1$</td><td>$[e_1, e_2] = e_2$</td><td>(310)(31)(1)</td><td><table><tr><td>α</td><td>0</td><td></td></tr><tr><td>$\psi(\alpha)$</td><td>6</td><td>4</td></tr></table></td></tr>
<tr><td>$\mathrm{g}_{3,1}$</td><td>$[e_2, e_3] = e_1$</td><td>(310)(310)(13)</td><td><table><tr><td>α</td><td></td></tr><tr><td>$\psi(\alpha)$</td><td>6</td></tr></table></td></tr>
<tr><td>$\mathrm{g}_{3,2}$</td><td>$[e_1, e_3] = e_1,$
$[e_2, e_3] = e_1 + e_2$</td><td>(320)(32)(0)</td><td><table><tr><td>α</td><td>1</td><td></td></tr><tr><td>$\psi(\alpha)$</td><td>4</td><td>3</td></tr></table></td></tr>
<tr><td>$\mathrm{g}_{3,3}$</td><td>$[e_1, e_3] = e_1,$
$[e_2, e_3] = e_2$</td><td>(320)(32)(0)</td><td><table><tr><td>α</td><td>1</td><td></td></tr><tr><td>$\psi(\alpha)$</td><td>6</td><td>3</td></tr></table></td></tr>
<tr><td>$\mathrm{g}_{3,4}(-1)$</td><td>$[e_1, e_3] = e_1,$
$[e_2, e_3] = -e_2$</td><td>(320)(32)(0)</td><td><table><tr><td>α</td><td>-1</td><td>1</td><td></td></tr><tr><td>$\psi(\alpha)$</td><td>5</td><td>4</td><td>3</td></tr></table></td></tr>
</table>

$\mathcal{L}$	**Commutators**		**Invariant function**
$g_{3,4}(a)$	$[e_1, e_3] = e_1,$ $[e_2, e_3] = ae_2,$	$(320)(32)(0)$	see table 1
$sl(2,\mathbb{C})$	$[e_1, e_3] = -2e_2,$ $[e_1, e_2] = e_1,$ $[e_2, e_3] = e_3$	$(3)(3)(0)$	see table 2

α	1	a	$\frac{1}{a}$	
$\psi(\alpha)$	4	4	4	3

α	-1	1	2	
$\psi(\alpha)$	5	3	1	0

$\mathcal{L}$	**Commutators**		**Invariant function**
$g_{3,4}(a)$	$[e_1, e_3] = e_1,$ $[e_2, e_3] = ae_2,$	(320)(32)(0)	
$sl(2, \mathbb{C})$	$[e_1, e_3] = -2e_2,$ $[e_1, e_2] = e_1,$ $[e_2, e_3] = e_3$	(3)(3)(0)	

α	1	a	$\frac{1}{a}$	
$\psi(\alpha)$	4	4	4	3

α	-1	1	2	
$\psi(\alpha)$	5	3	1	0

- **Theorem. (Classification of three–dimensional complex Lie algebras)** Let $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ be two three–dimensional complex Lie algebras. Then

$$\mathcal{L} \cong \widetilde{\mathcal{L}} \Leftrightarrow \psi\,\mathcal{L} = \psi\,\widetilde{\mathcal{L}}.$$

$\mathcal{L}$	**Commutators**		**Invariant function**
$\mathrm{g}_{3,4}(a)$	$[e_1, e_3] = e_1,$ $[e_2, e_3] = ae_2,$	$(320)(32)(0)$	see table below
$\mathrm{sl}(2, \mathbb{C})$	$[e_1, e_3] = -2e_2,$ $[e_1, e_2] = e_1,$ $[e_2, e_3] = e_3$	$(3)(3)(0)$	see table below

$\mathrm{g}_{3,4}(a)$:

α	1	a	$\frac{1}{a}$	
$\psi(\alpha)$	4	4	4	3

$\mathrm{sl}(2, \mathbb{C})$:

α	-1	1	2	
$\psi(\alpha)$	5	3	1	0

- **Theorem. (Classification of three–dimensional complex Lie algebras)** Let $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ be two three–dimensional complex Lie algebras. Then

$$\mathcal{L} \cong \widetilde{\mathcal{L}} \Leftrightarrow \psi\,\mathcal{L} = \psi\,\widetilde{\mathcal{L}}.$$

- **Theorem. (Classification of two–dimensional complex Jordan algebras)** Let $\mathcal{J}$ and $\widetilde{\mathcal{J}}$ be two two–dimensional complex Jordan algebras. Then

$$\mathcal{J} \cong \widetilde{\mathcal{J}} \Leftrightarrow \psi\,\mathcal{J} = \psi\,\widetilde{\mathcal{J}}.$$

Two–dimensional twisted cocycles

- $q = 2$

Two–dimensional twisted cocycles

- $q = 2$

- $\text{coh}_{(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3)}\ \mathcal{L} = Z^2\left(\mathcal{L}, \text{ad}, \begin{pmatrix} \beta_1 & \alpha_2 & \alpha_3 \\ \alpha_2 & \beta_3 & \alpha_1 \\ \alpha_3 & \alpha_1 & \beta_2 \end{pmatrix}\right)$

Two–dimensional twisted cocycles

- $q = 2$

- $\mathrm{coh}_{(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3)}\ \mathcal{L} = Z^2\left(\mathcal{L}, \mathrm{ad}, \begin{pmatrix} \beta_1 & \alpha_2 & \alpha_3 \\ \alpha_2 & \beta_3 & \alpha_1 \\ \alpha_3 & \alpha_1 & \beta_2 \end{pmatrix}\right)$

- $B \in C^2(\mathcal{L}, \mathcal{L})$

$$\begin{aligned} 0 &= \alpha_1 B(x,[y,z]) + \alpha_2 B(z,[x,y]) + \alpha_3 B(y,[z,x]) \\ &\quad + \beta_1[x, B(y,z)] + \beta_2[z, B(x,y)] + \beta_3[y, B(z,x)]. \end{aligned}$$

Two–dimensional twisted cocycles

- $q = 2$

- $\operatorname{coh}_{(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3)} \mathcal{L} = Z^2\left(\mathcal{L}, \operatorname{ad}, \begin{pmatrix} \beta_1 & \alpha_2 & \alpha_3 \\ \alpha_2 & \beta_3 & \alpha_1 \\ \alpha_3 & \alpha_1 & \beta_2 \end{pmatrix}\right)$

- $B \in C^2(\mathcal{L}, \mathcal{L})$

$$\begin{aligned} 0 &= \alpha_1 B(x,[y,z]) + \alpha_2 B(z,[x,y]) + \alpha_3 B(y,[z,x]) \\ &\quad + \beta_1[x,B(y,z)] + \beta_2[z,B(x,y)] + \beta_3[y,B(z,x)]. \end{aligned}$$

- **Theorem.** For any $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \in \mathbb{C}$ there exists $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that the subspace $\operatorname{coh}_{(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3)} \mathcal{L} \subset C^2(\mathcal{L},\mathcal{L})$ is equal to some of the following sixteen subspaces:

  1. $\operatorname{coh}_{(\alpha,0,0,\beta,0,0)} \mathcal{L}$; $\operatorname{coh}_{(\alpha,0,0,\beta,1,-1)} \mathcal{L}$; $\operatorname{coh}_{(\alpha,1,-1,\beta,0,0)} \mathcal{L}$; $\operatorname{coh}_{(\alpha,\beta,-\beta,\gamma,1,-1)} \mathcal{L}$
  2. $\operatorname{coh}_{(\alpha,0,0,\beta,1,0)} \mathcal{L}$; $\operatorname{coh}_{(\alpha,0,0,\beta,1,1)} \mathcal{L}$; $\operatorname{coh}_{(\alpha,\beta,-\beta,\gamma,1,0)} \mathcal{L}$; $\operatorname{coh}_{(\alpha,1,-1,\beta,1,1)} \mathcal{L}$
  3. $\operatorname{coh}_{(\alpha,1,0,\beta,0,0)} \mathcal{L}$; $\operatorname{coh}_{(\alpha,1,1,\beta,0,0)} \mathcal{L}$; $\operatorname{coh}_{(\alpha,1,0,\beta,\gamma,-\gamma)} \mathcal{L}$; $\operatorname{coh}_{(\alpha,1,1,\beta,1,-1)} \mathcal{L}$
  4. $\operatorname{coh}_{(\alpha,\beta,\gamma,\delta,1,0)} \mathcal{L}$; $\operatorname{coh}_{(\alpha,\beta+1,\beta-1,\gamma,1,1)} \mathcal{L}$; $\operatorname{coh}_{(\alpha,1,1,\beta,\gamma+1,\gamma-1)} \mathcal{L}$; $\operatorname{coh}_{(\alpha,\beta,\beta,\gamma,1,1)} \mathcal{L}$

Invariant functions

- $\dim \mathcal{L} = n$

- $\varphi\,\mathcal{L} : \mathbb{C} \to \{0, 1, 2, \ldots, n^2(n-1)/2\}$

$$\varphi\,\mathcal{L}(\alpha) = \dim \mathrm{coh}_{(1,1,1,\alpha,\alpha,\alpha)}\ \mathcal{L}$$

- $\varphi^0\,\mathcal{L} : \mathbb{C} \to \{0, 1, 2, \ldots, n^2(n-1)/2\}$

$$\varphi^0\,\mathcal{L}(\alpha) = \dim \mathrm{coh}_{(0,1,1,\alpha,1,1)}\ \mathcal{L}$$

Invariant functions

- $\dim \mathcal{L} = n$

- $\varphi\,\mathcal{L} : \mathbb{C} \to \{0, 1, 2, \ldots, n^2(n-1)/2\}$

$$\varphi\,\mathcal{L}(\alpha) = \dim \operatorname{coh}_{(1,1,1,\alpha,\alpha,\alpha)} \mathcal{L}$$

- $\varphi^0\,\mathcal{L} : \mathbb{C} \to \{0, 1, 2, \ldots, n^2(n-1)/2\}$

$$\varphi^0\,\mathcal{L}(\alpha) = \dim \operatorname{coh}_{(0,1,1,\alpha,1,1)} \mathcal{L}$$

$$\boxed{\mathcal{L} \cong \widetilde{\mathcal{L}} \;\Rightarrow\; \varphi\,\mathcal{L} = \varphi\,\widetilde{\mathcal{L}} \text{ and } \varphi^0\,\mathcal{L} = \varphi^0\,\widetilde{\mathcal{L}}}$$

- $\mathfrak{g}_{4,2}(a):$ $[e_1, e_4] = ae_1,\ [e_2, e_4] = e_2,\ [e_3, e_4] = e_2 + e_3$
  $a \neq 0, \pm 1, -2$

- $\mathfrak{g}_{4,2}(a)$: $[e_1, e_4] = ae_1,\ [e_2, e_4] = e_2,\ [e_3, e_4] = e_2 + e_3$
  $a \neq 0, \pm 1, -2$

α	1	a	$\frac{1}{a}$	
$\psi\, \mathfrak{g}_{4,2}(a)(\alpha)$	6	5	5	4

α	1	
$\psi^0\, \mathfrak{g}_{4,2}(a)(\alpha)$	1	0

- $\mathfrak{g}_{4,2}(a)$: $[e_1, e_4] = ae_1,\ [e_2, e_4] = e_2,\ [e_3, e_4] = e_2 + e_3$
  $a \neq 0, \pm 1, -2$

α	1	a	$\frac{1}{a}$	
$\psi\, \mathfrak{g}_{4,2}(a)(\alpha)$	6	5	5	4

α	1	
$\psi^0\, \mathfrak{g}_{4,2}(a)(\alpha)$	1	0

α	$1+a$	$\frac{2}{a}$	
$\varphi\, \mathfrak{g}_{4,2}(a)(\alpha)$	13	13	12

α	2	$1+a$	$1+\frac{1}{a}$	
$\varphi^0\, \mathfrak{g}_{4,2}(a)(\alpha)$	3	1	1	0

- $\mathfrak{g}_{4,2}(a)$: $[e_1, e_4] = ae_1,\ [e_2, e_4] = e_2,\ [e_3, e_4] = e_2 + e_3$
  $a \neq 0, \pm 1, -2$

α	1	a	$\frac{1}{a}$	
$\psi\, \mathfrak{g}_{4,2}(a)(\alpha)$	6	5	5	4

α	1	
$\psi^0\, \mathfrak{g}_{4,2}(a)(\alpha)$	1	0

α	$1+a$	$\frac{2}{a}$	
$\varphi\, \mathfrak{g}_{4,2}(a)(\alpha)$	13	13	12

α	2	$1+a$	$1+\frac{1}{a}$	
$\varphi^0\, \mathfrak{g}_{4,2}(a)(\alpha)$	3	1	1	0

- **Theorem. (Classification of four–dimensional complex Lie algebras)**
  Let $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ be two four–dimensional complex Lie algebras. Then

$$\mathcal{L} \cong \widetilde{\mathcal{L}} \Leftrightarrow \psi\, \mathcal{L} = \psi\, \widetilde{\mathcal{L}} \quad \text{and} \quad \varphi\, \mathcal{L} = \varphi\, \widetilde{\mathcal{L}}.$$

Contractions of Lie algebras

One–parametric continuous contractions of Lie algebras

Contractions of Lie algebras

One–parametric continuous contractions of Lie algebras

- $U : (0, 1\rangle \to GL(\mathcal{L})$ continuous

Contractions of Lie algebras

One–parametric continuous contractions of Lie algebras

- $U : (0,1\rangle \rightarrow GL(\mathcal{L})$ continuous
- $U(\varepsilon) \in GL(\mathcal{L}),\ 0 < \varepsilon \leq 1$

Contractions of Lie algebras

One–parametric continuous contractions of Lie algebras

- $U : (0, 1\rangle \to GL(\mathcal{L})$ continuous
- $U(\varepsilon) \in GL(\mathcal{L}),\ 0 < \varepsilon \leq 1$

$$[x, y]_0 = \lim_{\varepsilon \to 0+} U(\varepsilon)^{-1}[U(\varepsilon)x, U(\varepsilon)y]$$

Contractions of Lie algebras

One–parametric continuous contractions of Lie algebras

- $U : (0, 1\rangle \rightarrow GL(\mathcal{L})$ continuous
- $U(\varepsilon) \in GL(\mathcal{L}),\ 0 < \varepsilon \leq 1$

$$[x, y]_0 = \lim_{\varepsilon \to 0+} U(\varepsilon)^{-1}[U(\varepsilon)x, U(\varepsilon)y]$$

- $\mathcal{L} \rightarrow \mathcal{L}_0$

Contractions of Lie algebras

One–parametric continuous contractions of Lie algebras

- $U : (0, 1\rangle \to GL(\mathcal{L})$ continuous
- $U(\varepsilon) \in GL(\mathcal{L}),\ 0 < \varepsilon \leq 1$

$$[x, y]_0 = \lim_{\varepsilon \to 0+} U(\varepsilon)^{-1}[U(\varepsilon)x, U(\varepsilon)y]$$

- $\mathcal{L} \to \mathcal{L}_0$
- $\dim \operatorname{der} \mathcal{L} < \dim \operatorname{der} \mathcal{L}_0$

Contractions of Lie algebras

One–parametric continuous contractions of Lie algebras

- $U : (0,1\rangle \to GL(\mathcal{L})$ continuous
- $U(\varepsilon) \in GL(\mathcal{L}),\ 0 < \varepsilon \leq 1$

$$[x,y]_0 = \lim_{\varepsilon \to 0+} U(\varepsilon)^{-1}[U(\varepsilon)x, U(\varepsilon)y]$$

- $\mathcal{L} \to \mathcal{L}_0$
- $\dim \operatorname{der} \mathcal{L} < \dim \operatorname{der} \mathcal{L}_0 \ \Leftrightarrow\ \psi\, \mathcal{L}(1) < \psi\, \mathcal{L}_0(1)$

Contractions of Lie algebras

One–parametric continuous contractions of Lie algebras

- $U : (0,1\rangle \to GL(\mathcal{L})$ continuous
- $U(\varepsilon) \in GL(\mathcal{L}),\ 0 < \varepsilon \leq 1$

$$[x,y]_0 = \lim_{\varepsilon \to 0+} U(\varepsilon)^{-1}[U(\varepsilon)x, U(\varepsilon)y]$$

- $\mathcal{L} \to \mathcal{L}_0$
- $\dim \operatorname{der} \mathcal{L} < \dim \operatorname{der} \mathcal{L}_0 \ \Leftrightarrow\ \psi\,\mathcal{L}(1) < \psi\,\mathcal{L}_0(1)$
- $\mathcal{L} \to \mathcal{L}_0 \ \Rightarrow\ \psi\,\mathcal{L} \leq \psi\,\mathcal{L}_0$ and $\psi\,\mathcal{L}(1) < \psi\,\mathcal{L}_0(1)$.

Contractions of Lie algebras

One–parametric continuous contractions of Lie algebras

- $U : (0,1\rangle \to GL(\mathcal{L})$ continuous
- $U(\varepsilon) \in GL(\mathcal{L}),\ 0 < \varepsilon \leq 1$

$$[x,y]_0 = \lim_{\varepsilon \to 0+} U(\varepsilon)^{-1}[U(\varepsilon)x, U(\varepsilon)y]$$

- $\mathcal{L} \to \mathcal{L}_0$
- $\dim \operatorname{der} \mathcal{L} < \dim \operatorname{der} \mathcal{L}_0 \ \Leftrightarrow\ \psi\,\mathcal{L}(1) < \psi\,\mathcal{L}_0(1)$
- $\mathcal{L} \to \mathcal{L}_0 \ \Rightarrow\ \psi\,\mathcal{L} \leq \psi\,\mathcal{L}_0$ and $\psi\,\mathcal{L}(1) < \psi\,\mathcal{L}_0(1)$.
- **Theorem.** **(Contractions of three–dimensional complex Lie algebras)** Let $\mathcal{L}$ and $\mathcal{L}_0$ be two three–dimensional complex Lie algebras. Then there exists one–parametric continuous contraction $\mathcal{L} \to \mathcal{L}_0$ if and only if

$$\psi\,\mathcal{L} \leq \psi\,\mathcal{L}_0 \quad \text{and} \quad \psi\,\mathcal{L}(1) < \psi\,\mathcal{L}_0(1).$$

Contractions of Lie algebras

One–parametric continuous contractions of Lie algebras

- $U : (0,1\rangle \to GL(\mathcal{L})$ continuous
- $U(\varepsilon) \in GL(\mathcal{L}),\ 0 < \varepsilon \leq 1$

$$[x,y]_0 = \lim_{\varepsilon \to 0+} U(\varepsilon)^{-1}[U(\varepsilon)x, U(\varepsilon)y]$$

- $\mathcal{L} \to \mathcal{L}_0$
- $\dim \operatorname{der} \mathcal{L} < \dim \operatorname{der} \mathcal{L}_0 \;\Leftrightarrow\; \psi\,\mathcal{L}(1) < \psi\,\mathcal{L}_0(1)$
- $\mathcal{L} \to \mathcal{L}_0 \;\Rightarrow\; \psi\,\mathcal{L} \leq \psi\,\mathcal{L}_0$ and $\psi\,\mathcal{L}(1) < \psi\,\mathcal{L}_0(1)$.
- **Theorem.(Contractions of three–dimensional complex Lie algebras)** Let $\mathcal{L}$ and $\mathcal{L}_0$ be two three–dimensional complex Lie algebras. Then there exists one–parametric continuous contraction $\mathcal{L} \to \mathcal{L}_0$ if and only if

$$\psi\,\mathcal{L} \leq \psi\,\mathcal{L}_0 \quad \text{and} \quad \psi\,\mathcal{L}(1) < \psi\,\mathcal{L}_0(1).$$

- **Theorem.(Contractions of two–dimensional complex Jordan algebras)** Let $\mathcal{J}$ and $\mathcal{J}_0$ be two two–dimensional complex Jordan algebras. Then there exists one–parametric continuous contraction $\mathcal{J} \to \mathcal{J}_0$ if and only if

$$\psi\,\mathcal{J} \leq \psi\,\mathcal{J}_0 \quad \text{and} \quad \psi\,\mathcal{J}(1) < \psi\,\mathcal{J}_0(1).$$

Graded contractions of Lie algebras

Graded contractions of Lie algebras

- identification of results

Graded contractions of Lie algebras

- identification of results
- nilpotent one–parametric Lie algebras

Graded contractions of Lie algebras

- identification of results
- nilpotent one–parametric Lie algebras
- $\mathcal{L}(a)$ $\quad [e_1,e_3]=e_5,\ [e_1,e_4]=-ae_8,\ [e_2,e_3]=e_7,\ [e_2,e_4]=e_6,$
  $[e_3,e_5]=e_8,\ [e_3,e_7]=e_6,\ 0\le |a|\le 1$

α	0	1	
$\psi^0(\alpha)\mathcal{L}(a)$	16	10	9

$a=0$

α	0	
$[\psi\,\mathcal{L}(0)](\alpha)$	23	21

$a=1$

α	0	-1	1	
$[\psi\,\mathcal{L}(1)](\alpha)$	22	21	20	19

$a=-1$

α	0	1	
$[\psi\,\mathcal{L}(-1)](\alpha)$	22	22	19

$a\neq 0,\pm 1$

α	0	1	$-a$	$-\frac{1}{a}$	
$[\psi\,\mathcal{L}(a)](\alpha)$	22	20	20	20	19

Graded contractions of Lie algebras

- identification of results
- nilpotent one–parametric Lie algebras
- $\mathcal{L}(a)$ $\quad [e_1,e_3]=e_5,\ [e_1,e_4]=-ae_8,\ [e_2,e_3]=e_7,\ [e_2,e_4]=e_6,$
  $[e_3,e_5]=e_8,\ [e_3,e_7]=e_6,\ 0\le |a|\le 1$

α	0	1	
$\psi^0(\alpha)\mathcal{L}(a)$	16	10	9

$a=0$

α	0	
$[\psi\,\mathcal{L}(0)](\alpha)$	23	21

$a=1$

α	0	-1	1	
$[\psi\,\mathcal{L}(1)](\alpha)$	22	21	20	19

$a=-1$

α	0	1	
$[\psi\,\mathcal{L}(-1)](\alpha)$	22	22	19

$a\neq 0,\pm 1$

α	0	1	$-a$	$-\frac{1}{a}$	
$[\psi\,\mathcal{L}(a)](\alpha)$	22	20	20	20	19

- for $a,b\neq 0,\pm 1,\ a\neq b,\ a\neq \frac{1}{b}$ we have $[\psi\,\mathcal{L}(a)](-a)=20\neq[\psi\,\mathcal{L}(b)](-a)=19\Rightarrow \mathcal{L}(a)\ncong\mathcal{L}(b)$

References

1. George F. Leger and Eugene M. Luks: *Generalized derivations of Lie algebras*, J. Alg. 228, 165-203, (2000)

2. Jonas T. Hartwig, Sergei D. Sivestrov: *Generalized derivations on algebras* (2003)

3. Agaoka Y., *An algorithm to determine the isomorphism classes of 4-dimensional complex Lie algebras* Linear Algebra & Appl. **345** (2002), 85–118

4. Nesterenko M., Popovych R.: *Contractions of low–dimensional Lie algebras*, (2006) arxiv: math-ph/0608018

5. Ancochea Bermúdez J., Campoamor-Stursberg R., García Vergnolle L. and Sánchez Hernández J., *Contractions d'algebres de Jordan en dimension 2* (2007) arxiv: math.RA/0701477

6. Hrivnák J., Novotný P., Patera J. and Tolar J., *Graded contractions of the Pauli graded* $sl(3,\mathbb{C})$. Linear Algebra & Appl. **418**, Issues 2–3 (2006), 498–550