ABCs of symplectic reduction

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Moment map

$$R_g : M \to M \qquad R_g^* \omega = \omega$$

$$R_g \leftrightarrow \xi_X \qquad X \in \mathcal{G}$$

$$0 = \mathcal{L}_{\xi_X} \omega = i_{\xi_X} d\omega + d i_{\xi_X} \omega = d(i_{\xi_X} \omega)$$

$$i_{\xi_X} \omega =: -\alpha_X \qquad \Rightarrow \qquad d\alpha_X = 0$$

$$\alpha_{X+\lambda Y} = \alpha_X + \lambda \alpha_Y \qquad \lambda \in \mathbb{R}$$

$$\alpha_X = X^i \alpha_i \qquad \alpha_i := \alpha_{E_i}$$

$$R_g^* \alpha_X = \alpha_{\mathrm{Ad}\,_g X} \qquad \text{i.e.} \qquad \mathcal{L}_{\xi_X} \alpha_Y = \alpha_{[X,Y]}$$

Globally hamiltonian action: $\alpha_X = dP_X$.

$$i_{\xi_X}\omega = -dP_X \qquad \text{i.e.} \qquad \xi_X f = \{P_X, f\}$$

$$P_X := X^i P_i \qquad\qquad P_i \equiv P_{E_i}$$

$$P_{X+\lambda Y} = P_X + \lambda P_Y$$

$$P : M \to \mathcal{G}^* \qquad\qquad \langle P(x), X\rangle := P_X(x),\ x \in M$$

$$P = P_i E^i$$

$$R_g^* P_X = P_{\mathrm{Ad}\,_g X} + k(g, X) \qquad\qquad k(g, X) \in \mathbb{R}$$

$$\xi_X P_Y \equiv \{P_X, P_Y\} = P_{[X,Y]} + \beta(X, Y)$$

Poisson action: equivariant $P =$ moment map

$$\beta(X, Y) = 0 \qquad\qquad k(g, X) = 0$$

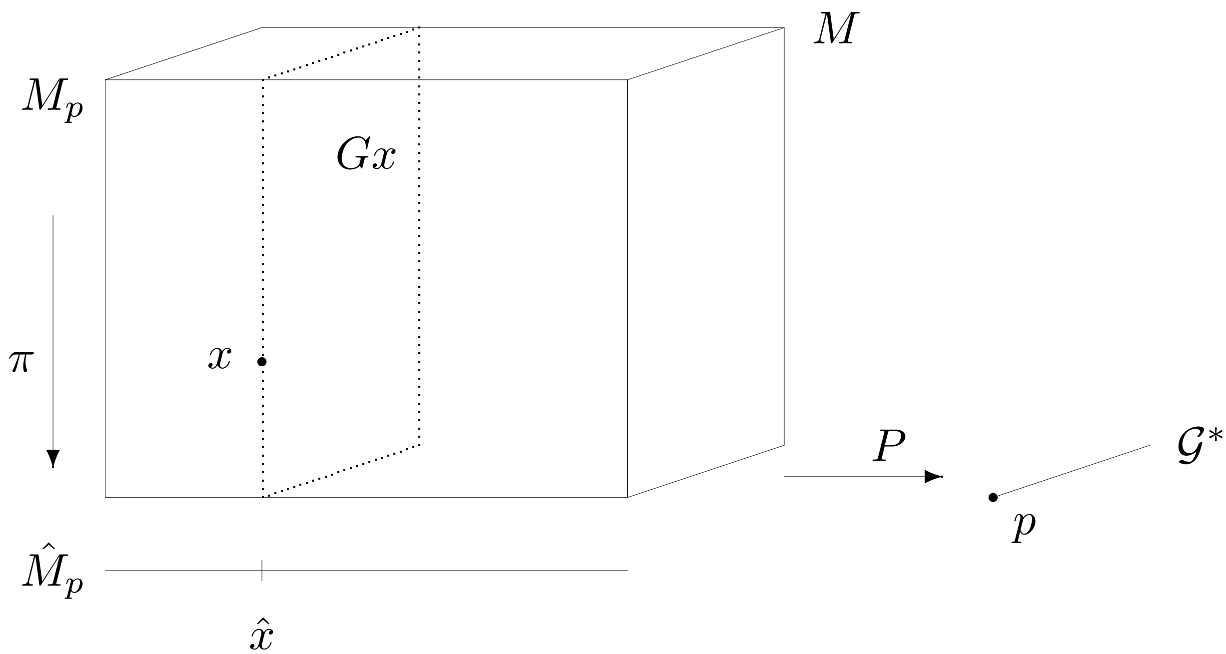
$$\begin{array}{ccc} M & \xrightarrow{\ P\ } & \mathcal{G}^* \\ {\scriptstyle R_g}\downarrow & & \downarrow{\scriptstyle \mathrm{Ad}\,_g^*} \\ M & \xrightarrow[\ P\ ]{} & \mathcal{G}^* \end{array}$$

In mechanics:

if G preserves (M, ω, H), then $\dot{P} \equiv \dot{P}_i E^i = 0$

Symplectic reduction

Free, Poisson action of G on (M, ω)



$$M_p := \{x \in M \mid P(x) = p\}$$

$$P_i(x) = p_i \equiv \text{ const.}, \quad i = 1, \ldots, \dim G$$

$$R_g M_p = M_{\mathrm{Ad}\,_g^* p}$$

$$R_g M_p = M_p \qquad\qquad g \in G_p = \text{ stabilizer of } p$$

$$\pi : M_p \to \hat{M}_p \qquad\qquad \text{principal } G_p \text{ bundle}$$

$$R_g^* \tilde{\omega} = \tilde{\omega} \qquad g \in G_p,\ \tilde{\omega} \equiv \omega|_{M_p}$$

$$\tilde{\omega}(w,\ .\ ) = 0 \quad \text{for any vertical vector } w$$

$\tilde{\omega}$ is G_p-invariant and vertical, so pull-back of $\hat{\omega}$ on $\hat{M}_p$:

$$\tilde{\omega} = \pi^* \hat{\omega}$$

It is closed and non-degenerate $\Rightarrow$ symplectic.

$$(M, \omega) \ \mapsto \ (\hat{M}_p, \hat{\omega}) \quad = \text{reduced phase space}$$

$$(M, \omega, H) \ \mapsto \ (\hat{M}_p, \hat{\omega}, \hat{H}) \quad = \text{reduced hamiltonian system}$$

Example 1:

Original hamiltonian system:

$\mathbb{R}^6[\mathbf{r},\mathbf{p}]$, $\omega = d\mathbf{p}.\wedge d\mathbf{r}$, $H(\mathbf{r},\mathbf{p}) = \mathbf{p}^2/2m + mgz$

Symmetry group: translations in x, y directions (lifted action)

Reduced hamiltonian system:

$\mathbb{R}^2[z,p]$, $\hat{\omega} = dp \wedge dz$, $\hat{H}(z,p) = \frac{p^2}{2m} + mgz$

Example 2:

Original hamiltonian system:

Phase space $\mathbb{R}^{12}[\mathbf{r}_1,\mathbf{r}_2,\mathbf{p}_1,\mathbf{p}_2] \equiv T^*\mathbb{R}^6[\mathbf{r}_1,\mathbf{r}_2]$

$$H(\mathbf{r}_1,\mathbf{r}_2,\mathbf{p}_1,\mathbf{p}_2) = \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} + U(|\mathbf{r}_1 - \mathbf{r}_2|)$$

Symmetry group: lifted translations

$$R_{\mathbf{a}} : (\mathbf{r}_1,\mathbf{r}_2,\mathbf{p}_1,\mathbf{p}_2) \mapsto (\mathbf{r}_1 + \mathbf{a},\mathbf{r}_2 + \mathbf{a},\mathbf{p}_1,\mathbf{p}_2)$$

Reduced hamiltonian system:

$\hat{M}_p = \mathbb{R}^6[\mathbf{r},\mathbf{p}]$, $\hat{\omega} = d\mathbf{p}.\wedge d\mathbf{r}$, $\hat{H} = \hat{H}(\mathbf{r},\mathbf{p}) = \frac{\mathbf{p}^2}{2\mu} + U(r)$

Example 3:

Further reduction by rotations:

Symmetry group: lifted rotations

$$\mathbf{r} \mapsto A\mathbf{r} \qquad \mathbf{p} \mapsto A\mathbf{p} \qquad A \in SO(3)$$

Reduced hamiltonian system:

$\hat{M}_p = \mathbb{R}^2(r, p_r)$, $\hat{\omega} = dp_r \wedge dr$, $\hat{H}(r, p_r) = \frac{p_r^2}{2\mu} + \frac{L^2}{2\mu r^2} + U(r)$

Example 4:

$\mathbb{C}P^n$ from $\mathbb{C}^{n+1}$ via $U(1)$ reduction

Original: $\mathbb{C}^{n+1}$ with canonical symplectic form

$$\omega = -\frac{i}{2}\; d\bar{z}^a \wedge dz^a$$

Reduced: $\mathbb{C}P^n$ with Fubini-Study symplectic form

$$\tilde{\omega} = -\frac{i}{2}\; h_{\bar{k}j}\; d\bar{w}^k \wedge dw^j \qquad h_{\bar{k}j} = \frac{(1 + \bar{w}w)\delta^{kj} - w^k \bar{w}^j}{(1 + \bar{w}w)^2}$$

THANK YOU