On the spin chains

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1. Ground State of the Heisenberg Ferromagnet
2. Elliptic Functions
3. Integrals of Motion of the Spin Chain with Weierstrass Function Interaction

1. Ground State of the Heisenberg Ferromagnet

Spins on a lattice, Hilbert space $C^2 \otimes C^2 \otimes \ldots \otimes C^2$

$$H = \sum_{<i,j>} J_{ij} \tfrac{1}{4}\left(1 - \vec{\sigma}_i \cdot \vec{\sigma}_j\right) \quad , \quad \vec{\sigma}_i = (\sigma_i^x, \sigma_i^y, \sigma_i^z)$$

i,j lattice sites, <i,j> edges connecting sites i and j

$\sigma_i^x, \sigma_i^y, \sigma_i^z$ Pauli matrices acting on the spin i

Heisenberg, Dirac, Van Vleck – interaction of atoms

W. Heisenberg, Zs.Phys. **29**(1928), 619.
J.H. Van Vleck, *The Theory of Electric and Magnetic Susceptibilities*, Clarendon Press, Oxford, 1932.
D.C. Matis, *The Theory of Magnetism I. Statics and Dynamics*, Springer, Berlin, 1981.

$$H = \sum_{<i,j>} J_{ij} \tfrac{1}{4}\left(1 - \vec{\sigma}_i \cdot \vec{\sigma}_j\right)$$

$J_{ij} > 0$ ferromagnet $J_{ij} < 0$ antiferromagnet

Ground state of ferromagnet $\left|\psi_0\right\rangle = \left|\uparrow\uparrow\uparrow \cdots \uparrow\right\rangle$

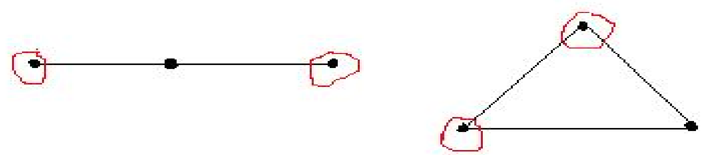
and its rotations and their linear combinations,
and no other states
Generally believed to be easy to prove, e.g.,
D. Mattis, Phys. Rev. **130**(1963), 76.
Published rigorous proof only (up to our knowledge)
J. Dittrich, V.I. Inozemtsev, Czech. J. Phys. **54**(2004), 775

Lemma. Let Γ be a connected graph with $N \geq 2$ vertices 1,2, ..., N. Then there exist two vertices $i_0 \neq j_0$ such that the graphs $\Gamma \backslash \{i_0\}$ and $\Gamma \backslash \{j_0\}$ are connected.

Proof by induction according to N.

N=2

N=3

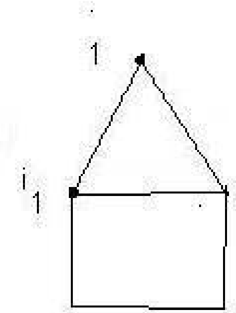


Assume validity for graphs up to $N \geq 2$ vertices, let Γ' be a graph of N+1 vertices, $\Gamma'' = \Gamma' \backslash \{1\}$.

1) Γ'' connected

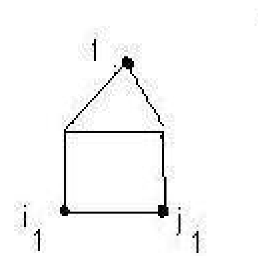
Let $\Gamma''\backslash\{i_1\}$, $\Gamma''\backslash\{j_1\}$ be connected.

a) $<1,i_1> \subset \Gamma' \Rightarrow \Gamma'\backslash\{1\}, \Gamma'\backslash\{j_1\}$ connected



b) $<1,i_1> \not\subset \Gamma'$, $<1,j_1> \not\subset \Gamma' \Rightarrow$

$\Gamma'\backslash\{i_1\}, \Gamma'\backslash\{j_1\}$ connected



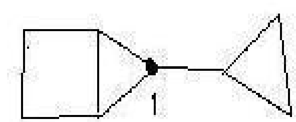
2) Γ'' disconnected

a) component with ≥ 2 vertices:

two can be removed, at least one, i,

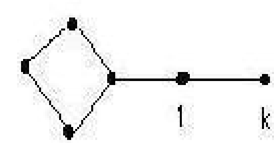
without breaking connection to 1,

i.e. $\Gamma'\backslash\{i\}$ remains connected



b) Component with one vertex k:

$\Gamma'\backslash\{k\}$ remains connected



Theorem. Let be |ψ> a ground state of the Hamiltonian

$$H = \sum_{<i,j>\subset\Gamma} J_{ij}\,\tfrac{1}{4}\left(1-\vec{\sigma}_i\cdot\vec{\sigma}_j\right) \quad , \quad J_{ij}>0$$

where Γ is a connected graph of N vertices, H |ψ> = 0. Then

a) $$\vec{S}^2\left|\psi\right\rangle = \frac{N}{2}\left(\frac{N}{2}+1\right)\left|\psi\right\rangle \quad , \quad \vec{S}=\frac{1}{2}\sum_{i=1}^{N}\vec{\sigma}_i \qquad \text{(maximal total spin)}$$

b) $$\left(1-\vec{\sigma}_i\cdot\vec{\sigma}_j\right)\left|\psi\right\rangle = 0 \qquad \text{for any i,j=1,…,N}$$

c) There exist complex numbers α_k and matrices u_k such that

$$\left|\psi\right\rangle = \sum_{k=1}^{N+1}\alpha_k \bigotimes_{i=1}^{N} u_k\left|\uparrow\right\rangle \qquad \text{(unique up to rotations and superpositions)}$$

Proof. Spectrum of $\vec{\sigma}_i \cdot \vec{\sigma}_j$ is {1,-3}

$$\langle \varphi | (1 - \vec{\sigma}_i \cdot \vec{\sigma}_j) | \varphi \rangle \geq 0$$

$$\langle \varphi | (1 - \vec{\sigma}_i \cdot \vec{\sigma}_j) | \varphi \rangle = 0 \quad \Leftrightarrow \quad (1 - \vec{\sigma}_i \cdot \vec{\sigma}_j) | \varphi \rangle = 0$$

$$b) \Rightarrow a) \Rightarrow c)$$

Proof of the Theorem by induction, assume validity for N. Let Γ be a connected graph with N+1 vertices, Γ\{1} and Γ\{N+1} be connected graphs according to the Lemma. By assumption,

$$\vec{\sigma}_i \cdot \vec{\sigma}_j | \psi \rangle = | \psi \rangle$$

for i,j = 2,...,N+1 and i,j = 1,...,N.

Unknown only

$$\vec{\sigma}_1 \cdot \vec{\sigma}_{N+1} | \psi \rangle = ?$$

$$[H,\vec{S}^2]=0$$

$$\left|\psi\right\rangle = \sum_S \left|\psi_S\right\rangle \quad , \quad \vec{S}^2\left|\psi_S\right\rangle = S(S+1)\left|\psi_S\right\rangle \quad , \quad H\left|\psi_S\right\rangle = 0$$

$$\vec{S}^2\left|\psi_S\right\rangle = \left[\left(\sum_{i=1}^{N}\tfrac{1}{2}\vec{\sigma}_i\right)^2 + 2\sum_{i=1}^{N}\tfrac{1}{4}\vec{\sigma}_i\cdot\vec{\sigma}_{N+1} + \tfrac{1}{4}\vec{\sigma}_{N+1}{}^2\right]\left|\psi_S\right\rangle$$

$$= \left[\frac{N}{2}\left(\frac{N}{2}+1\right) + \tfrac{1}{2}\vec{\sigma}_1\cdot\vec{\sigma}_{N+1} + \tfrac{1}{2}\left(N-1\right) + \tfrac{3}{4}\right]\left|\psi_S\right\rangle$$

$$= \tfrac{1}{4}\left(N^2+4N+1\right)\left|\psi_S\right\rangle + \tfrac{1}{2}\vec{\sigma}_1\cdot\vec{\sigma}_{N+1}\left|\psi_S\right\rangle$$

So $|\psi_S>$ is an eigenvector of $\vec{\sigma}_1\cdot\vec{\sigma}_{N+1}$, eigenvalues 1 and -3

We shall assume the second possibility and exclude it, this will complete the proof.

$$\vec{\sigma}_1 \cdot \vec{\sigma}_{N+1} \left| \psi_S \right\rangle = -3 \left| \psi_S \right\rangle$$

$$S(S+1) = \tfrac{1}{4}(N^2 + 4N - 5)$$

1) $N = 2n \, , n \geq 1 \, , S = k + \tfrac{1}{2} \, , k \geq 0$

$$(n-k)\underbrace{(n+k+2)}_{>2} = 2 \quad \textit{impossible}$$

2) $N = 2n+1 \, , n \geq 1 \, , S = k \geq 0$

$$k(k+1) = n(n+3) \quad , \quad k > n \quad , \quad k = n+m \quad , \quad m > 0$$

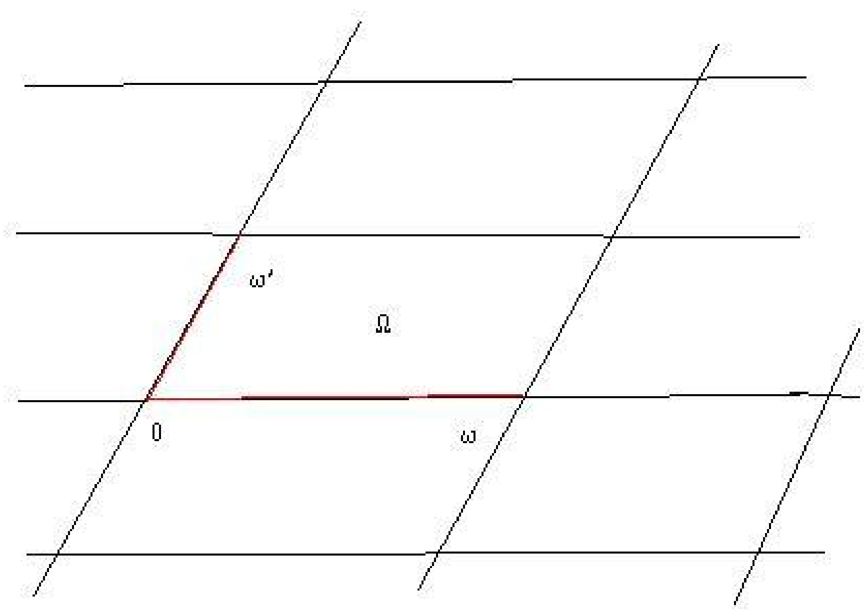
$$\underbrace{2n(m-1)}_{\geq 0} + \underbrace{m(m+1)}_{>0} = 0 \quad \textit{impossible}$$

The only possibility $\vec{\sigma}_1 \cdot \vec{\sigma}_{N+1} \left| \psi_S \right\rangle = \left| \psi_S \right\rangle$

Theorem is completely proved.

2. Elliptic Functions

Meromorphic functions in the complex plain
with two periods ω, ω', $\mathrm{Im}(\omega'/\omega)>0$



$$F(z+\omega) = F(z+\omega') =$$
$$= F(z)$$

Liouville theorems for elliptic functions

1) Σ residues = 0
   integrate along the boundary of fundamental parallelogram Ω

$$\varphi(z) = \frac{F'(z)}{F(z)}$$

2) Number of poles = number of zeroes
3) Regular elliptic function is constant
4) Number of poles of non-constant elliptic function ≥ 2

Weierstrass functions

$$\wp(z) = \frac{1}{z^2} + \sum_{m,m'}{}' \left\{ \frac{1}{(z - m\omega - m'\omega')^2} - \frac{1}{(m\omega + m'\omega')^2} \right\}$$

$\sum'$ over integer m, m' except of $m = m' = 0$

$$\varsigma(z) = \frac{1}{z} + \sum_{m,m'}{}' \left\{ \frac{1}{z - m\omega - m'\omega'} + \frac{1}{m\omega + m'\omega'} + \frac{z}{(m\omega + m'\omega')^2} \right\}$$

$$\varsigma'(z) = -\wp(z) \quad , \quad \frac{\sigma'(z)}{\sigma(z)} = \varsigma(z)$$

$\wp$ elliptic, ς and σ not elliptic

Many identities holds, you can prove your owns with the help of Liouville theorems if needed.

N.I. Akhiezer, *The elements of elliptic functions theory* (Nauka, Moscow, 1970; in Russian)
E.T.Whittaker, G.N. Watson, *A course of modern analysis* (Cambridge Univ. Press, 1952; Chapt. 20-21).

Further assumed ω=N (number of spins in a closed chain, ω'=i κ, κ>0.

3. Integrals of Motion for the Spin Chain with Weierstrass Function Interaction

$$P_{jk} = \tfrac{1}{4}(1 - \vec{\sigma}_j \cdot \vec{\sigma}_k)$$

interchanges the spin states at sites j and k, can be also considered as transposition. We can work in the representation of the permutation group without reference to the spins.

$$H = h_0 \sum_{\substack{j,k=1 \\ j \neq k}}^{N} \wp(j-k) P_{jk} \quad , \quad h_0 = const$$

$$J = \sum_{j\neq k\neq l\neq j} f(j-k)\, f(k-l)\, f(l-j) P_{jkl}$$

$P_{jkl} = P_{jk} P_{kl}$ cyclic permutation of (j,k,l), or

$$P_{jkl} = \tfrac{1}{16}\left[1 - \vec{\sigma}_j \cdot \vec{\sigma}_k - \vec{\sigma}_k \cdot \vec{\sigma}_l - i\vec{\sigma}_j \cdot (\vec{\sigma}_k \times \vec{\sigma}_l)\right]$$

$$f(x) = \frac{\sigma(x+\alpha)}{\sigma(x)\sigma(\alpha)} e^{-x\varsigma(\alpha)}$$, α is a parametr

$$[H, J] = 0$$

V.I. Inozemtsev, J. Stat. Phys. **59** (1990), 1143

$$J = -\tfrac{1}{2}\wp'(\alpha) J_0 + \wp(\alpha) J_1 - \tfrac{1}{2} J_2$$

$$J_0 = \sum_{j\neq k\neq l\neq j} P_{jkl}$$

$$J_1 = \sum_{j\neq k\neq l\neq j} P_{jkl}\left[\varsigma(j-k) + \varsigma(k-l) + \varsigma(l-j)\right] P_{jkl}$$

$$J_2 = \tfrac{1}{3} \sum_{j \neq k \neq l \neq j} \Big\{ \wp'(j-k) + \wp'(k-l) + \wp'(l-j) + $$
$$+ 2\big[\varsigma(j-k) + \varsigma(k-l) + \varsigma(l-j)\big]^3 \Big\} P_{jkl}$$

α arbitrary $\Rightarrow$

$[H, J_0] = 0 \quad , \quad [H, J_1] = 0 \quad , \quad [H, J_2] = 0$

Trivially $\quad [J_0, P_{jk}] = 0 \quad , \quad [J_0, J_1] = 0 \quad , \quad [J_0, J_2] = 0$

Non - trivially using the properties of elliptic functions

$[J_1, J_2] = 0$

J_0, J_1, J_2 are linearly independent

J. Dittrich, V.I. Inozemtsev, manuscript in preparation

4. Conclusions

1) For the ferromagnetic chain a ground state is

$$|\psi_0\rangle = |\uparrow\uparrow\ldots\uparrow\rangle$$

Any other ground state is obtained from it by rotations and linear combinations. A detailed rigorous proof done.

2) For spin chain with Weierstrass p-function interaction 3 integrals of motion are known, their commutativity and linear independence was proved.
See V.I. Inozemtsev, Lett. Math. Phys. **36** (1996), 55 for the existence of integrals of motion depending on the higher cyclic permutations (products of more P_{jk}).

Acknowledgement

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