The Tresse theorem and its application to nonlinear Schrödinger equation

T. Czyżycki

Plan:

1. Basic notations and definitions
2. The Tresse theorem
3. Examples
4. Application the Tresse theorem to the Schrödinger equation
5. Conclusions

1. Basic notations and definitions

u denotes a real function $u(x_1, ..., x_n)$,

u_{x_i} denotes the partial derivative $\frac{\partial u}{\partial x_i}$,

$\underset{k}{u}$ denotes the set of all partial derivatives of the order k of a function u,

D_i denotes the operator of the full differentiation over x_i.

$\underset{m}{X}$ denotes the extension of the m - th order of a vector field X to the space $(x_1, x_2, ...x_n, u, \underset{1}{u}, \underset{2}{u}, ..., \underset{m}{u})$ and its defined by the formula:

$$\underset{m}{X} = X + \sum_{p=1}^{m} \zeta^{i_1,\ldots,i_p} \partial_{u_{x_{i_1,\ldots,i_p}}},$$

where coefficients $\zeta^{i_1,\ldots,i_p}$ are defined by:

$$\zeta^{i_1,\ldots,i_p} = D_{i_1,\ldots,i_p}(\eta - u_{x_k}\xi^k) + u_{x_{i_1},\ldots,x_{i_p},x_{i_k}} \cdot \xi^k,$$

where the summation is over k;

$(i_1, i_2, \ldots, i_m)$ are fixed,

$$X = \xi^i(x, u, \underset{1}{u}, \ldots, \underset{s}{u})\partial_{x_i} + \eta(x, u, \underset{1}{u}, \ldots, \underset{s}{u})\partial_u$$

Definition 1 *Let G be a Lie group of transformations with the parameter $a \in \mathbb{R}$,*
$f, g \in G,\ x \in \mathbb{R}^n,\ u = u(x_1, ..., x_n)$ *and*
$\widetilde{x} = f(x, u, a),\ \widetilde{u} = g(x, u, a).$

a) A function $F(x, u)$ is called an invariant of G iff :

$$\forall_{a \in \mathbb{R}}\ F(\widetilde{x}, \widetilde{u}) = F(x, u).$$

b) An expression $F(x, u, \underset{1}{u}, \underset{2}{u}, ... \underset{m}{u})$ is called a differential invariant (of the m–th order) of the group G iff :

$$\forall_{a \in \mathbb{R}}\ F(\widetilde{x}, \widetilde{u}, \underset{1}{\widetilde{u}}, \underset{2}{\widetilde{u}}, ... \underset{m}{\widetilde{u}}) = F(x, u, \underset{1}{u}, \underset{2}{u}, ... \underset{m}{u}).$$

c) The general (or universal) differential invariant of the m-th order is the set of all differential invariants from the order zero to the order m inclusive,

d) A maximal set of functionally independent invariants of the order $r \leqslant m$ of a Lie group G is called a functional basis of the m–th order differential invariants of G,

e) Q is called an operator of the invariant differentiation, if for any differential invariant F of the group G the expression QF is also the differential invariant of the group G.

2. The Tresse theorem (1894)

For a given Lie group G with r parameters, acting in the space (x, u), $x \in V \subset \mathbb{R}^n$, $u : V \to \mathbb{R}, (\mathbb{C})$ **there exists a finite basis** of functionally independent invariants and exist operators of the invariant differentiation Q_j such that arbitrary fixed order invariant of G can be obtained in a finite number of invariant differentiations and functional operations on invariants from the basis.

This finite basis includes in the general differential invariant of the minimal order $s \geqslant 1$ such that:

$$r = rank[\xi(x,u), \eta(x,u), \zeta^1(x,u,\underset{1}{u}), ..., \zeta^{s-1}(x,u,\underset{1}{u}, ..., \underset{s-1}{u})] \tag{1}$$

Operators of the invariant differentiation are defined by:

$$Q_j = \lambda_j^i(x,u,\underset{1}{u}, ..., \underset{s}{u})D_i, \tag{2}$$

where $\lambda_j = [\lambda_j^i]$ satisfies the condition:

$$\underset{s}{X}_{\nu}\lambda_j = \lambda_j^i D_i(\xi_\nu), \tag{3}$$

Remark 1 If a Lie group G acts in the space $(x_1, ..., x_n, u_1, ..., u_k) \in \mathbb{R}^{n+k}$, then the number of elements in a basis of the m–th order general invariant is given by the formula

$$R(m) = n + k \cdot \binom{n+m}{n} - r_m, \qquad (4)$$

where r_m is a rank of the matrix of coefficients of the m–th prolongation of operators X_ν.

3. Examples

a) Consider the group of rotations in $\mathbb{R}^3$:

$$\begin{cases} \widetilde{x} = x\cos a - y\sin a \\ \widetilde{y} = x\sin a + y\cos a \quad , \qquad r = 1,\ s = 1, \\ \widetilde{u} = u \end{cases}$$

with infinitesimal generator $X = -y\partial_x + x\partial_y$.

Invariants of the order zero satisfy the equation $X\omega = 0$ and they are

$$\omega_{01} = u,\ \omega_{02} = x^2 + y^2$$

$$\underset{1}{X} = -y\partial_x + x\partial_y - u_y\partial_{u_x} + u_x\partial_{u_y}$$

and system (3) has the form:

$$x\lambda_y - y\lambda_x - u_y\lambda_{u_x} + u_x\lambda_{u_y} = \lambda^1 \cdot [0,1]^T + \lambda^2 \cdot [-1,0]^T.$$

Hence

$$Q_1 = u_x D_x + u_y D_y, \quad Q_2 = -u_y D_x + u_x D_y.$$

The basis of a general invariant of the first order consists of four elements:

$$u, \; x^2 + y^2, \; u_x^2 + u_y^2 = Q_1(\omega_{01}), \; xu_x + yu_y = \frac{1}{2}Q_1(\omega_{02})$$

b) The Lorentz group in $(x, y, u) \in \mathbb{R}^3$ with the generator

$$X = y\partial_x + x\partial_y, \qquad r = 1,\ s = 1.$$

The base of invariants of the order zero has the form:

$$u,\ x^2 - y^2.$$

The first order basic invariants are: $u_x^2 - u_y^2, \quad xu_x + yu_y.$

The invariant differentiation operators are:

$$Q_1 = xD_x + yD_y, \qquad Q_2 = u_xD_x - u_yD_y,$$

and the first order basic invariants can be obtained from the zeroth order ones.

4. Application of the Tresse theorem to the Schrödinger equation

Consider the NSE of the form:

$$i\psi_t + \psi_{xx} + |\psi|^2\psi = 0. \tag{5}$$

It admits the infinite dimensional Lie algebra of symmetry, but if we consider the system:

$$\begin{cases} i\psi_t + \psi_{xx} + |\psi|^2\psi = 0 \\ -i\psi_t^* + \psi_{xx}^* + |\psi|^2\psi^* = 0, \end{cases} \tag{6}$$

then we obtain only the 5–dimensional, solvable Lie algebra of symmetry of this system:

$$X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = \psi \partial_\psi - \psi^* \partial_{\psi^*},$$

$$X_4 = t\partial_x + \frac{i}{2} x (\psi \partial_\psi - \psi^* \partial_{\psi^*}),$$

$$X_5 = 2t\partial_t + x\partial_x - \psi \partial_\psi - \psi^* \partial_{\psi^*}$$

We find the differential invariants of this algebra.

$$R(0) = 2 + 2 \cdot \binom{2+0}{2} - 4 = 0$$

$$R(1) = 2 + 2 \cdot \binom{2+1}{2} - 5 = 3$$

$$\omega_1 = \frac{\psi_x}{|\psi|\psi} + \frac{\psi_x^*}{|\psi|\psi^*}, \quad \omega_2 = \frac{\psi_t}{|\psi|^2\psi} - i \cdot \left(\frac{\psi_x}{|\psi|\psi}\right)^2,$$

$$\omega_3 = \frac{\psi_t^*}{|\psi|^2\psi^*} + i \cdot \left(\frac{\psi_x^*}{|\psi|\psi^*}\right)^2.$$

$$\omega_4 = \frac{1}{|\psi|^2\psi^2}(\psi_{xx}\psi - \psi_x^2), \qquad \omega_4^* = \overline{\omega_4},$$

$$\omega_5 = \frac{\psi_{tx}\psi - \psi_x\psi_t}{|\psi|^3\psi^2} + \frac{i(\psi\psi_x^* - \psi_x\psi^*)}{|\psi^5|} \cdot \left(\frac{\psi_{xx}}{\psi} - \frac{\psi_x^2}{\psi^2}\right),$$

$$\omega_5^* = \overline{\omega_5},$$

$$\omega_6 = \frac{1}{|\psi|^6} \cdot [\psi\psi^* \cdot D_t(\Omega) + i(\psi\psi_x^* - \psi_x\psi^*) \cdot D_x(\Omega)],$$

$$\text{where} \quad \Omega = \frac{\psi_t}{\psi} - i\left(\frac{\psi_x}{\psi}\right)^2,$$

$$\omega_6^* = \overline{\omega_6}$$

Operators of invariant differentiation are:

$$Q_1 = \frac{1}{|\psi|} D_x, \quad Q_2 = \frac{1}{|\psi|^2} D_t + \frac{i(\psi\psi_x^* - \psi_x\psi^*)}{|\psi|^4} D_x$$

and it appears that all second order differential invariants can be obtained from the first order ones.

We have the invariant form of the studied NSE:

$$i\omega_2 + \omega_4 + 1 = 0$$

5. Conclusions

1) Using the Tresse theorem one can find the basis of invariants and state whether or not some invariant is fundamental

2) The conservation laws are fundamental invariants of the Lie group of symmetry of PDE

3) NSE is not a fundamental invariant of its Lie group of symmetry