Canonical decomposition of the harmonic polynomials and Fourier transforms of $\mathbf{SO}(d)$-finite measures

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Preliminaries on spherical harmonics

We adopt the following notations and definitions.
S^{d-1} — the unit sphere in the euclidean space of dimension $d \geqslant 3$;
α is a constant defined by $d = 2\alpha + 2$
$(x|y)$ is the scalar product of $x,\ y \in \mathbb{R}^d$, $r^2 = (x|x)$ — the length^2;
$\Delta f = \sum_{j=1}^{d} \frac{\partial^2 f}{\partial x_j^2}$ — the Laplacian in $\mathbb{R}^d$.
$\mathcal{P}^l = \mathcal{P}^l(\mathbb{R}^d)$ - the space of homogeneous polynomials of degree l,
$\mathcal{H}^l \subset \mathcal{P}^l(\mathbb{R}^d)$ — the kernel of Δ in $\mathcal{P}^l$, i.e. the space of harmonic and homogeneous polynomials of degree l, with

$$\dim \mathcal{H}^l = \frac{2(l+\alpha)\Gamma(2\alpha+l)}{\Gamma(l+1)\Gamma(2\alpha+1)}.$$

The *surface spherical harmonics* of order l are defined as restrictions of elements from $\mathcal{H}^l$ to the unit sphere S^{d-1}.

Theorem : The canonical decomposition

The space $\mathcal{P}^l$ *decomposes orthogonally* $\mathcal{P}^l = \oplus_{k=0}^{[l/2]} r^{2k}\mathcal{H}^{l-2k}$ *and the decomposition is invariant with respect to the action of* $K = \mathbf{SO}(d)$. *Thus any polynomial* $P \in \mathcal{P}^l$ *may be written as*

$$P = \sum_{k=0}^{[l/2]} r^{2k} h_{l-2k}(P), \qquad \textit{with} \quad h_{l-2k}(P) \in \mathcal{H}^{l-2k}, \tag{1}$$

and the harmonic components of P *in the formula* (1) *are given*

$$h_{l-2k}(P) = \sum_{j=0}^{[l/2]-k} e_j^l(k) r^{2j} \Delta^{k+j} P \tag{2}$$

where

$$e_j^l(k) = (-1)^j \frac{(\alpha + l - 2k)\Gamma(\alpha + l - 2k - j)}{4^{k+j} k! j! \Gamma(\alpha + l + 1 - k)}.$$

The canonical decomposition, Theorem c.d.

In particular, set $x = |x|\xi$, take an arbitrary $\eta \in S^{d-1}$ and let l be a nonnegative integer, then

$$(x \mid \eta)^l = 2^{-l}\Gamma(\alpha)\Gamma(l+1)|x|^l \sum_{k=0}^{[l/2]} \frac{(\alpha + l - 2k)}{k!\Gamma(\alpha + l + 1 - k)} C_{l-2k}^{\alpha}((\xi \mid \eta)). \tag{3}$$

Here $C_l^{\alpha}(t)$ is the so called Gegenbauer polynomial defined by

$$C_l^{\alpha}(t) = \sum_{j=0}^{[l/2]} (-1)^j \frac{\Gamma(\alpha + l - j)}{\Gamma(\alpha)\Gamma(j+1)\Gamma(l+1-2j)} (2t)^{l-2j} \tag{4}$$

The zonal polynomial

The group $\mathbf{SO}(d)$ acts irreducibly in $\mathcal{H}^l$ for each l, and there exists a unique, up to a scalar multiple, polynomial in $\mathcal{H}^l$ which is invariant under the action of the isotropy subgroup $K_\eta \simeq \mathbf{SO}(d-1)$ of a point $\eta \in S^{d-1}$. This polynomial is called **the zonal polynomial** of degree l with pole at η and is given by

$$Z_\eta^l(x) = |x|^l \big[C_l^\alpha(1)\big]^{-1} C_l^\alpha((\,\xi \mid \eta\,)), \qquad x = |x|\xi. \tag{5}$$

The general form of an expansion
The profile function

We call a function f defined on the unit sphere S^{d-1} *a zonal function (relative to a point $\eta \in S^{d-1}$)* if it is invariant with respect to the isotropy group K_η of η. Any such function is in fact a function of one variable, namely the scalar product $(\xi \mid \eta)$ and as such can be written in the form

$$f(\xi) = \phi((\xi \mid \eta)),\ \xi \in S^{d-1},$$

where the function ϕ defined on the unit interval $[-1, 1] \subset \mathbb{R}$ will be called **the profile function of f**.

The general form of an expansion

The Fourier–Laplace expansion

Every square integrable function on the sphere can be written as a series of spherical harmonics (the Fourier–Laplace expansion), which for zonal functions reduces to

$$f(\xi) = \sum_{m=0}^{\infty} f_m Z_\eta^m(\xi), \text{ where } f_m = \dim \mathcal{H}^m \int_{S^{d-1}} f(\xi) Z_\eta^m(\xi)\, d\sigma(\xi). \tag{6}$$

The following result shows how the coefficients f_k of the expansion can be expressed in terms of the coefficients of the Taylor expansion of the profile function ϕ.

The general form of an expansion

Theorem

Assume $\phi : [-1,1] \to \mathbb{C}$ *has the Taylor expansion* $\phi(t) = \sum_{m=0}^{\infty} \frac{\phi^{(m)}(0)}{m!} t^m$ *absolutely convergent on the interval* $[-1,1]$ *and let* $f(\xi) = \phi((\xi \mid \eta))$ *be the zonal function on the sphere* S^{d-1} *corresponding to* ϕ*. Then the spherical Fourier expansion of* $f(\xi)$ *is given by*

$$f(\xi) = \Gamma(\alpha+1) \sum_{m=0}^{\infty} f_m \dim \mathcal{H}^m Z_\eta^m(\xi), \tag{7}$$

where the coefficients of the expansion can be expressed by

$$f_m = \sum_{k=0}^{\infty} \frac{\phi^{(m+2k)}(0)}{2^{m+2k} k! \Gamma(\alpha + m + k + 1)}. \tag{8}$$

The Bessel function

The Bessel function of the first kind are defined by the power series convergent in the whole complex plane $\mathbb{C}$

$$J_\nu(t) = \left(\frac{t}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\nu+1)} \left(\frac{t}{2}\right)^{2k}$$

where ν is a complex number satisfying $\mathrm{Re}\,\nu > -1$. The so called spherical Bessel functions are given by

$$j_\nu(t) = \Gamma(\nu+1)\left(\frac{t}{2}\right)^{-\nu} J_\nu(t).$$

Applications : 1) The plane wave expansion

As an immediate application of the Theorem 1 we get

Corollary

For arbitrary unit vectors $\xi, \eta \in S^{d-1} \subset \mathbb{R}^d$ and $u \in \mathbb{R}_+$ the plane wave $e^{iu(\xi|\eta)}$ admits the following expansion

$$e^{iu(\xi|\eta)} = \sum_{m=0}^{\infty} i^m \dim \mathcal{H}^m \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+m+1)} \left(\frac{u}{2}\right)^m j_{\alpha+m}(u) Z_\eta^m(\xi).$$

The series converges absolutely for each fixed value of $u \in \mathbb{R}_+$ and uniformly with respect to $\xi, \eta \in S^{d-1}$.

Applications : 2) The expansion of the Poisson kernel

The Poisson kernel for the unit ball in $\mathbb{R}^d$ is given by

$$P(r\eta;\xi) = \frac{1 - r^2}{(1 - 2r(\xi \mid \eta) + r^2)^{d/2}}, \qquad \text{where } 0 \leqslant r < 1.$$

For fixed r it is a zonal function with a pole η and from the Theorem 1 one can obtain a new proof of the following classical expansio

$$P(r\eta;\xi) = \sum_{m=0}^{\infty} \dim \mathcal{H}^m r^m Z_\eta^m(\xi).$$

Corollary

Let a function f on the sphere S^{d-1} be a finite sum of surface spherical harmonics of different degrees, $f = \sum_{k=1}^{p} Y_k$, where Y_k are restrictions to the sphere of elements in $\mathcal{H}^{n_k}$. Then the Fourier transform of the measure $f(\eta)\, d\sigma(\eta)$ is given by the formula

$$\int_{S^{d-1}} e^{i(x|\eta)} f(\eta)\, d\sigma(\eta) = \sum_{k=1}^{p} \left(\frac{i}{2}\right)^{n_k} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+n_k+1)} j_{\alpha+n_k}(|x|)\, Y_k(x)$$

In the terminology adopted in harmonic analysis functions satisfying the conditions of the Corollary are called $\mathbf{SO}(d)$-finite, since their $\mathbf{SO}(d)$-translates span a finite dimensional subspace.

Theorem (F. J. Gonzalez Vieli, 2000; A.Bezubik, A.Strasburger, 2005)

If $P \in \mathcal{P}^l$, then the Fourier transform of the measure $P(\xi) d\sigma(\xi)$ with support on the unit sphere S^{d-1} is given by the following equivalent formulae

$$\int_{S^{d-1}} e^{i(x|\eta)} P(\eta)\, d\sigma(\eta) = \tag{9}$$

$$\left(\frac{i}{2}\right)^l \sum_{k=0}^{[l/2]} \frac{(-1)^k 2^{2k} \Gamma(\alpha+1)}{\Gamma(\alpha+l+1-2k)} j_{\alpha+l-2k}(|x|) h_{l-2k}(P)(x) = \tag{10}$$

$$= \left(\frac{i}{2}\right)^l \sum_{k=0}^{[l/2]} \frac{(-1)^k \Gamma(\alpha+1)}{k! \Gamma(\alpha+l+1-k)} j_{\alpha+l-k}(|x|)(\Delta^k P)(x) \tag{11}$$

with $h_{l-2k}(P)$ denoting the harmonic components of P as in the Eqn. (1).

Preliminaries on spherical harmonics in complex space

We identify n-dimentional complex vector space with the even dimensional Euclidean space $\mathbb{R}^{2n}$, $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, setting $z_j = x_j + iy_j$, for $j = 1, \ldots, n$, and $(z_1, ..., z_n) \to (x_1, ..., x_n, y_1, ..., y_n)$.
Hermitian inner product in the space $\mathbb{C}^n$ is given by

$$(z|\xi) = \sum_{i=1}^{n} z_i \overline{\xi_i}, \qquad z = (z_1, ..., z_n),\ \xi = (\xi_1, ..., \xi_n)$$

and the corresponding norm

$$|\xi|^2 = (\xi|\xi) = \sum_{i=1}^{n} |\xi_i|^2.$$

G — special unitary group of degree n of matrices which satisfy

$$G = \mathbf{SU}(n) = \{A\ :\ A\overline{A}^T = I,\ A \in M_n(\mathbb{C}),\ \det A = 1\},$$

where $\overline{A}^T = A^*$ is the Hermitian conjugation.

The unit sphere and the isotropy group

The unit sphere $S = \{z \in \mathbb{C}^n \mid |z|^2 = 1\}$ in $\mathbb{C}^n$ can be identified with the unit sphere $S^{2n-1} \subset \mathbb{R}^{2n}$ in the real Euclidean space $\mathbb{R}^{2n}$:

$$S \ni (z_1, ..., z_n) \longrightarrow (x_1, ..., x_n, y_1, ..., y_n) \in S^{2n-1}.$$

The sphere S can be realized as a coset space (a homogeneous space) of the group G in the following way. Let K denote the isotropy group of $e_n \in \mathbb{C}^n$ in G, i.e.

$$K = \{A \in G|\ Ae_n = e_n\}.$$

It is easy to see that if $A \in \mathbf{SU}(n)$ then $Ae_n = e_n$ if and only if A splits into the following block form

$$A = \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix}$$

where $R \in \mathbf{SU}(n-1),\ 1 \in \mathbb{C}$, and 0 stands for a $(n-1)$-column (resp. row) zero vector. This implies

$$S^{2n-1} \simeq S = G/K = \mathbf{SU}(n)\,/\,\mathbf{SU}(n-1)\,.$$

Identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ leads to an embedding of $\mathbf{SU}(n)$ in $\mathbf{SO}(2n)$:

$$\mathbf{SU}(n) \ni A = [a_{ij}] = [\alpha_{ij} + i\beta_{ij}] \mapsto \begin{pmatrix} (\alpha_{ij}) & (\beta_{ij}) \\ -(\beta_{ij}) & (\alpha_{ij}) \end{pmatrix} \in \mathbf{SO}(2n)\,.$$

Polynomial in $\mathbb{C}^n$

Every complex valued polynomial on $\mathbb{R}^{2n}$ homogeneous of degree m can be written using the multi-index notation in the following form

$$P(x_1, ..., x_n, y_1, ..., y_n) = \sum p_{kl} x^k y^l,$$

where $p_{kl} \in \mathbb{C}^n$, $x^k = x_1^{k_1} ... x_n^{k_n}$, $y^l = x_1^{l_1} ... y_n^{l_n}$ with multi-indices $k = (k_1, ..., k_n)$, $l = (l_1, ..., l_n) \in \mathbb{Z}_+^n$, $|k| + |l| = \sum k_i + \sum l_i = m$. Since $x_j = \frac{1}{2}(z_j + \bar{z}_j)$, $y_j = \frac{1}{2i}(z_j - \bar{z}_j)$ we may write it in the complex multi-index notation

$$P(z, \overline{z}) = \sum q_{\alpha\beta} z^\alpha \overline{z}^\beta,$$

where $\alpha = (\alpha_1, ..., \alpha_n)$, $\beta = (\beta_1, ..., \beta_n) \in \mathbb{Z}_+^n$, $\alpha_1 + \alpha_2 + ... + \alpha_n + \beta_1 + ... + \beta_n = m$ and $q_{\alpha\beta} \in \mathbb{C}$.

We shall denote by $\mathcal{P}^m = \mathcal{P}^m(\mathbb{C}^n)$ the complex vector space consisting of polynomial functions defined on $\mathbb{C}^n$, which are homogeneous of degree m. There are certain distinguished subspaces of $\mathcal{P}^m$ consisting of the so called bi-homogeneous polynomials which are defined as follows.

Definition

A polynomial $P(z, \overline{z})$ is called bi-homogeneous of degree (p, q), if

$$P(uz, \overline{uz}) = u^p \overline{u}^q P(z, \overline{z}), \qquad \text{for each } u \in \mathbb{C}.$$

We have the following formula

$$\langle\, P \mid Q \,\rangle := \int_{S^{2n-1}} P(\xi) \overline{Q(\xi)}\, d\sigma(\xi)$$

for the inner product in $\mathcal{P}^m(\mathbb{C}^n)$.

The Laplacian

A polynomial $P \in \mathcal{P}^m(\mathbb{C}^n)$ is harmonic if $\Delta P = 0$, where of course

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + \sum_{j=1}^{n} \frac{\partial^2}{\partial y_j^2}$$

is the Laplacian. We recall that Δ can be written in the complex form by using complex derivatives

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right); \qquad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

as

$$\Delta = 4\sum_{j=1}^{n} \frac{\partial^2}{\partial z_j \partial \overline{z_j}}.$$

The space $\mathcal{H}^m = \mathcal{P}^m \cap \ker \Delta$ consists of harmonic and homogeneous of degree m polynomials, while $\mathcal{H}^{(p,q)} = \mathcal{P}^{(p,q)} \cap \ker \Delta$ is the space of harmonic and bi-homogeneous of degree (p, q) polynomials. The restrictions of harmonic polynomials to the unit sphere are called spherical (surface) harmonics.

Since the action of $\mathbf{SU}(n)$ on $\mathcal{P}^m$ commutes with the Laplacian. It leaves $\mathcal{H}^m$ invariant. Moreover, each $\mathcal{P}^{(p,q)}$ is also invariant under this action.

Theorem (The complex case)

The space $\mathcal{P}^m$ decomposes orthogonaly as

$$\mathcal{P}^m(\mathbb{R}^{2n}) = \bigoplus_{p,q\geqslant 0, p+q=m} \mathcal{P}^{(p,q)}(\mathbb{C}^n)$$

and further

$$\begin{array}{rcll} \mathcal{P}^{(p,q)}(\mathbb{C}^n) & = & \mathcal{H}^{(p,q)} + r^2\mathcal{H}^{(p-1,q-1)} + \dots & + \\ & & +r^{2p}\ \mathcal{H}^{(0,q-p)} & , \quad p < q \\ & & +r^{2q}\ \mathcal{H}^{(p-q,0)} & , \quad p > q \end{array}$$

Thus from the group theoretic point of view the above given decomposition is the decomposition into irreducible representations of the group $\mathbf{SU}(n)$. It can be shown [5] that each of the space $\mathcal{H}^{(p,q)}$ is irreducible under the action of $\mathbf{SU}(n)$ and for different bi-degree the actions are inequivalent.

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