Role of entropies in quantum communication; Nilanjana Datta

Problems and References

Exercise 1 Use the monotonicity of the relative entropy under CPTP maps to show the following:

1. If $\sigma_{AB'} := (\mathrm{id}_A \otimes \Lambda)\rho_{AB}$, where $\Lambda : \mathcal{D}(\mathcal{H}_B) \to \mathcal{D}(\mathcal{H}_{B'})$ is a CPTP map, then $I(A : B')_\sigma = I(A; B)_\rho$.
2. $S(A_1A_2|B_1B_2) \leq S(A_1|B_1) + S(A_2|B_2)$

Exercise 2 Use strong subadditivity of the von Neumann entropy to prove

$$(i)\ S(A|BC)_\rho \leq S(A|B)_\rho \quad (ii)\ I(A:B)_\rho \leq I(A:BC)_\rho$$

Exercise 3 Use the Typical Subspace Theorem to establish that for any $\delta > 0$ and n large enough, the ensemble average fidelity satisfies the bound $\overline{F}_n \geq 1 - 2\delta$, for the encoding and decoding maps of the quantum data compression scheme for a memoryless source discussed in *Lecture 1.*

Exercise 4 Prove that the Holevo capacity $\chi^*(\mathcal{N})$ of a quantum channel $\mathcal{N}$ is superadditive, i.e., $\chi^*(\mathcal{N}_1 \otimes \mathcal{N}_2) \geq \chi^*(\mathcal{N}_1) + \chi^*(\mathcal{N}_2)$ for two quantum channels $\mathcal{N}_1$ and $\mathcal{N}_2$.

Exercise 5 Prove that the coherent information $I_{coh}(\mathcal{N}, \rho)$ of a quantum channel $\mathcal{N}$ for an input state ρ can be expressed in the form

$$I_{coh}(\mathcal{N}, \rho) = S(\mathcal{N}(\rho)) - S(\mathcal{N}^c(\rho)),$$

where $\mathcal{N}^c$ denotes the complementary channel.

Exercise 6 A quantum channel $\mathcal{N}$ is said to be *degradable* if there exists a CPTP map $\mathcal{T}$ such that $\mathcal{T} \circ \mathcal{N}(\rho) = \mathcal{N}^c(\rho)$. Using Exercise 5 and Exercise 1(2.) prove that if $\mathcal{N}_1$ and $\mathcal{N}_2$ are degradable, then the coherent information for the channel $\mathcal{N}_1 \otimes \mathcal{N}_2$ for any input state ρ_{12} satisfies the bound

$$I_{coh}(\mathcal{N}_1 \otimes \mathcal{N}_2, \rho_{12}) \leq I_{coh}(\mathcal{N}_1, \rho_1) + I_{coh}(\mathcal{N}_2, \rho_2),$$

where ρ_1 and ρ_2 are the reduced states of ρ_{12}. Use this result to establish that the quantum capacity of a degradable channel $\mathcal{N}$ is given by the following single-letter expression:

$$Q(\mathcal{N}) = \max_{\rho} I_{coh}(\mathcal{N}),$$

where the maximization is over all possible inputs to the channel.

Exercise 7 Prove that $H_{max}(A|B)_{\Phi^{(m)}} = -\log m$ for a maximally entangled state of Schmidt rank m: $|\Phi^{(m)}\rangle = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} |i_A\rangle |i_B\rangle$.

Exercise 8 Uhlmann's theorem for the fidelity of two states can also be expressed as follows: Given two states ρ and σ, for any fixed purification ψ of ρ we have

$$F(\rho, \sigma) = \max_{|\phi\rangle} |\langle \psi | \phi \rangle|, \tag{1}$$

where the maximization is over all possible purifications $|\phi\rangle$ of the state σ. Using (1) prove the following: If $\mathcal{U}(\rho_{AC}) := (I \otimes U)\rho_{AC}(I_A \otimes U^\dagger)$, where $U : \mathcal{H}_C \to \mathcal{H}_{C'} \otimes \mathcal{H}_E$ denotes the Stinespring isometry of a quantum channel $\Lambda : \mathcal{D}(\mathcal{H}_\mathcal{C}) \to \mathcal{D}(\mathcal{H}_{\mathcal{C}'})$, and $\Phi_{AC'}$ is a maximally entangled state, then there exists a state χ_E such that

$$F((\mathrm{id} \otimes \Lambda)\rho_{AC}, \Phi_{AC'}) = F(\mathcal{U}(\rho_{AC}), \Phi_{AC'} \otimes \chi_E).$$

Use this to establish that

$$H_{\min}(A|C)_\rho = -\log d_A \max_{C \to C'E} \max_{\omega_E} \log F^2(\mathcal{U}(\rho_{AC}), \Phi_{AC'} \otimes \omega_E),$$

where the first maximization is over all possible isometries $U : \mathcal{H}_C \to \mathcal{H}_{C'} \otimes \mathcal{H}_E$. [Hint: recall that in *Lecture 2* we proved that

$$H_{\min}(A|C)_\rho = -\log d_A \max_{\substack{\Lambda : \mathcal{D}(\mathcal{H}_\mathcal{C}) \to \mathcal{D}(\mathcal{H}_{\mathcal{C}'}) \\ CPTP}} \log F^2((\mathrm{id}_A \otimes \Lambda)\rho_{AC}, \Phi_{AC'}).]$$

Exercise 9 Use the above exercise to prove that if ρ_{ABC} is a purification of ρ_{AB}, then the following *duality* relation holds:

$$H_{\max}(A|B)_\rho = -H_{\min}(A|C)_\rho.$$

[Hint: Note that $H_{\max}(A|B)_\rho = \max_{\sigma_B} \log F^2(\rho_{AB}, I_A \otimes \sigma_B)$]

Remark: *The duality relation has also been proved to be valid for smoothed entropies, i.e. for a pure state* ρ_{ABC}*,*

$$H^{\varepsilon}_{\max}(A|B)_{\rho} = -H^{\varepsilon}_{\min}(A|C)_{\rho}.$$

The smooth entropies are defined as follows:

$$H^{\varepsilon}_{\min}(A|C)_{\rho} := \max_{\overline{\rho}_{AC} \in B^{\varepsilon} \in \rho_{AC}} H_{\min}(A|C)_{\overline{\rho}},$$

$$H^{\varepsilon}_{\max}(A|B)_{\rho} := \min_{\overline{\rho}_{AB} \in B^{\varepsilon} \in \rho_{AB}} H_{\max}(A|B)_{\overline{\rho}},$$

where

$$B^{\varepsilon}(\rho) := \{\overline{\rho} \in \mathcal{B}(\mathcal{H}) : \overline{\rho} \geq 0, Tr\,\overline{\rho} \leq 1, \tilde{F}(\rho, \overline{\rho}) \geq 1 - \varepsilon\},$$

with $\tilde{F}(\rho, \overline{\rho}) := F(\rho, \overline{\rho} + \sqrt{(1 - Tr\,\rho)(1 - Tr\,\overline{\rho})}.$

Some Relevant References:

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3. M. Tomamichel, "A Framework for Non-Asymptotic Quantum Information Theory," PhD thesis, Department of Physics, ETH Zurich; arXiv:1203.2142

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5. N. Datta, "Min- and Max- Relative Entropies and a New Entanglement Monotone," IEEE Transactions on Information Theory, vol. 55, p.2816-2826, June 2009.

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8. N. Datta and F. Leditzky, "Second-order asymptotics for source coding, dense coding and pure-state entanglement conversions," arXiv:1403.2543.

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