Role of Entropies in Quantum Communication

LECTURE III

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- *Consider transmission of information through quantum channels in the the one-shot setting:*
  - *transmission of quantum information…..*

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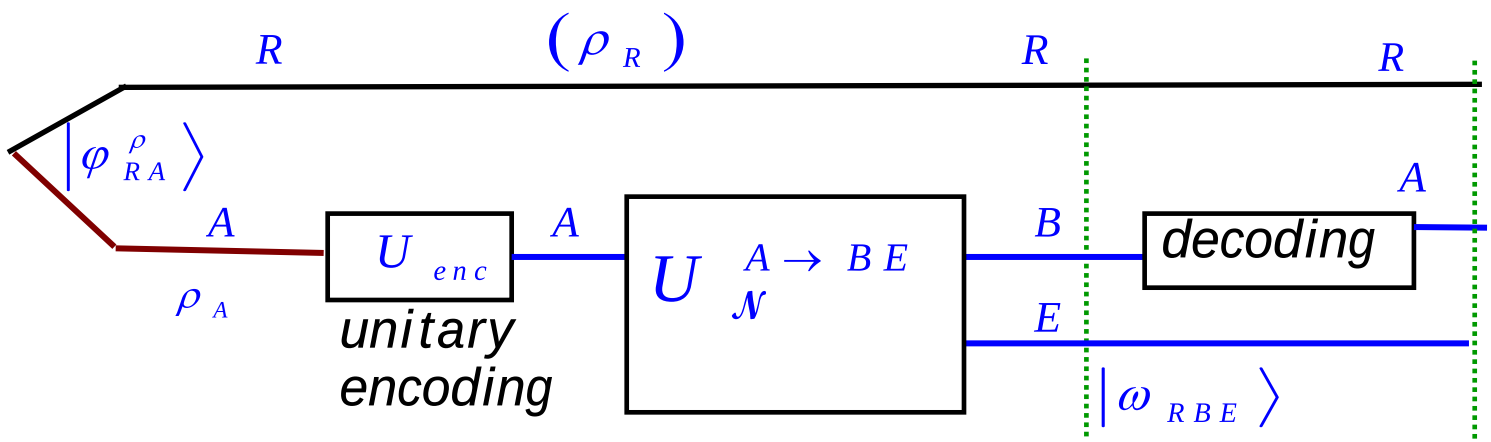
In the last lecture we saw that:

For transmission of quantum information through a noisy channel $\mathcal{N}$ in the one-shot setting (up to an error ε), require:

$$\omega_{RE} \overset{\varepsilon}{\approx} \rho_R \otimes \sigma_E$$

(state before decoding)



i.e., the state of the reference system R is (approxly.) decoupled from the state of the environment E of $\mathcal{N}$.

- *In this lecture: We shall make use of decoupling in evaluating the quantum capacity of a channel*

Mathematical Tool

(1) Choi-Jamilkowski (C-J) Isomorphism

Quantum operations ⟷ Positive operators

Let $\mathcal{H}_R \simeq \mathcal{H}_A$ with orthonormal basis $\{|i\rangle\}_{i=1}^{d}$

$$\Phi_{RA} := |\Phi_{RA}\rangle\langle\Phi_{RA}|; \quad |\Phi_{RA}\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d}|i\rangle|i\rangle \in \mathcal{H}_R \otimes \mathcal{H}_A$$

maximally entangled state (MES)

- **Choi state of a quantum operation $\Lambda^{A\to B}$:**

$$\sigma_{RB} := \left(\mathrm{id}_R \otimes \Lambda^{A\to B}\right)\Phi_{RA} \in \mathcal{P}(\mathcal{H}_R \otimes \mathcal{H}_A)$$

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History (decoupling)

First explicit use of decoupling in the Q.Info. Theory literature:

- *M.Horodecki, J.Oppenheim & A.Winter*

  (state merging)

- *A.Abeysinghe, I.Devetak, P.Hayden, A.Winter*

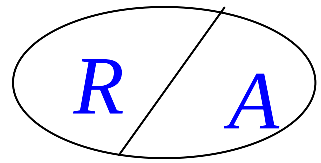
  (coherent state merging - FQSW)

- *P.Hayden, M.Horodecki, J.Yard & A.Winter*

  (quantum info transmission)

"asymptotic memoryless setting"

Decoupling Theorem



gives conditions under which 2 subsystems of a bipartite system can become almost uncorrelated (decoupled)

Rough idea: Initially ρ_{RA} : *possibly correlated*

- *Consider a unitary evolution of system* A *alone:*

$$(I \otimes U)\rho_{RA}(I \otimes U^{\dagger})$$

- *Then consider an arbitrary quantum operation on* A

e.g. $\Lambda \equiv \Lambda^{A \to E}$: $\omega_{RE} := (\mathrm{id} \otimes \Lambda)\Big((I \otimes U)\rho_{RA}(I \otimes U^{\dagger})\Big)$

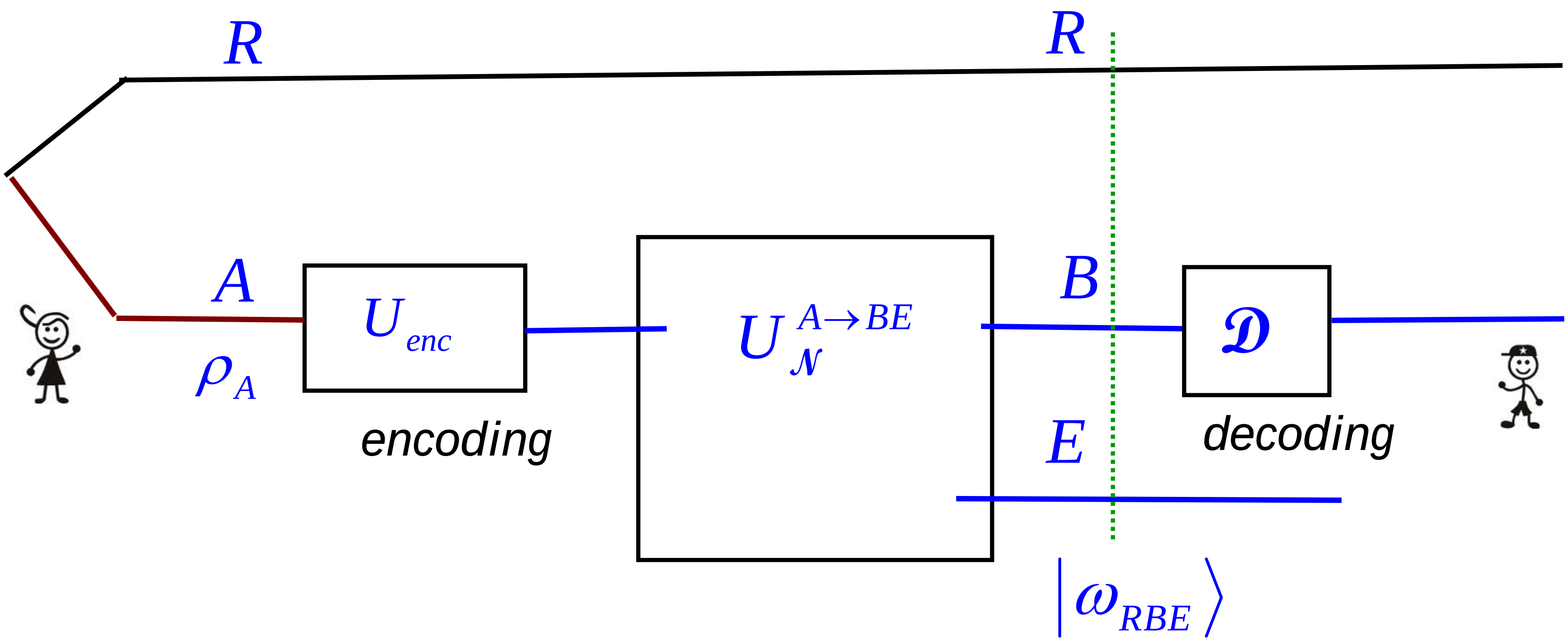
- *Decoupling theorem provides a bound on the distance of a typical resulting state from a decoupled state:*

  i.e., bound on $\| \omega_{RE} - \rho_R \otimes \sigma_E \|_1$

Decoupling Theorem

$$\rho_{RA} \to (I \otimes U)\rho_{RA}(I \otimes U^\dagger) \to (\mathrm{id} \otimes \Lambda)\big((I \otimes U)\rho_{RA}(I \otimes U^\dagger)\big) \equiv \omega_{RE};$$

- *For quantum info transmission through a noisy channel:*



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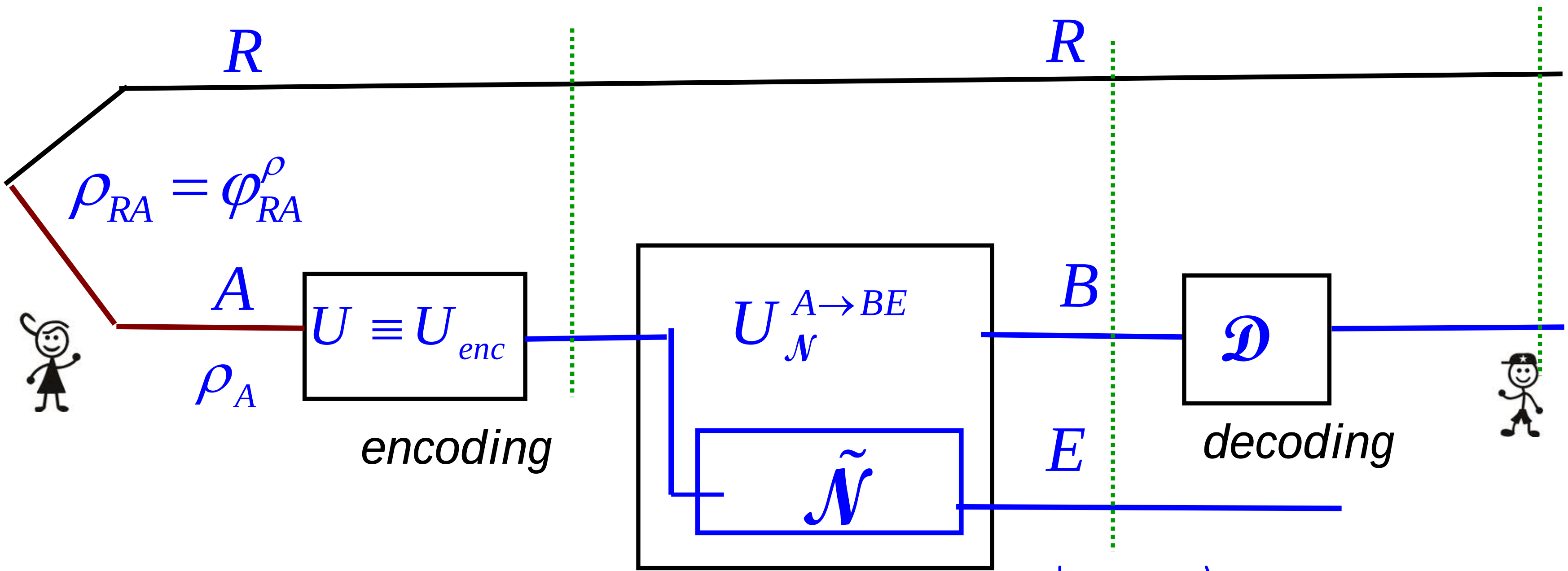
Decoupling Theorem

$$\rho_{RA} \to (I \otimes U)\rho_{RA}(I \otimes U^{\dagger}) \to (\mathrm{id} \otimes \Lambda)\big((I \otimes U)\rho_{RA}(I \otimes U^{\dagger})\big) \equiv \omega_{RE};$$

- *For quantum info transmission through a noisy channel:*



$|\omega_{RBE}\rangle$

$\Lambda^{A \to E} \equiv \tilde{\mathcal{N}}^{A \to E}$: *complementary channel*

$$\omega_{RE} = (\mathrm{id} \otimes \tilde{\mathcal{N}})\big((I \otimes U)\rho_{RA}(I \otimes U^{\dagger})\big)$$

History (decoupling) contd.

- *Decoupling theorem provides a bound on*

$$\| \omega_{RE} - \rho_R \otimes \sigma_E \|_1 \qquad \omega_{RE} \equiv \omega_{RE}(U)$$

One-shot setting : decoupling theorems in which the bound on $\| \omega_{RE} - \rho_R \otimes \sigma_E \|_1$ is given in terms of min/max entropies:

- [Berta], [Buscemi &ND], [Berta, Christandl, Renner], [ND, Hsieh]
- [Dupuis];
- [Dupuis, Berta, Wullschleger, Renner]

(decoupling condition expressed in terms of Choi state of the quantum operation)

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- **One-shot decoupling theorem:** *[Dupuis et al]*

Let $\varepsilon > 0;$ $\Lambda^{A\to E}:$ *(quantum operation)*

$$\sigma_{A'E} = (\mathrm{id}_{A'} \otimes \Lambda)\Phi_{A'A}$$ *Choi state of* Λ

Then for any state ρ_{RA},

$$\int \| (\mathrm{id}\otimes\Lambda)((I\otimes U)\rho_{RA}(I\otimes U^\dagger)) - \rho_R \otimes \sigma_E \|_1 \, dU$$

$$[= \omega_{RE}(U)]$$

$$\overset{\varepsilon}{\leq} 2^{-\frac{1}{2}H_{\min}^{\varepsilon}(A|R)_\rho - \frac{1}{2}H_{\min}^{\varepsilon}(A'|E)_\sigma}$$

$\int \bullet \, dU$: *integral over the Haar measure on the full unitary group over* $\mathcal{H}_A$

$$\sigma_E = \mathrm{Tr}_{A'}\sigma_{A'E}$$

- **One-shot decoupling theorem:** *[Dupuis et al]*

Let $\varepsilon > 0;$ $\Lambda^{A\to E} \equiv \tilde{\mathcal{N}}^{A\to E};$ *(quantum operation)*

$$\sigma_{A'E} = (\mathrm{id}_{A'} \otimes \Lambda)\Phi_{A'A}$$ *C-J state of* Λ

Then for any state ρ_{RA},

$$\int \| (\mathrm{id} \otimes \Lambda)((I \otimes U)\rho_{RA}(I \otimes U^{\dagger})) - \rho_R \otimes \sigma_E \|_1 \, dU \overset{\varepsilon}{\leq} 2^{-\frac{1}{2}H_{\min}^{\varepsilon}(A|R)_{\rho} - \frac{1}{2}H_{\min}^{\varepsilon}(A'|E)_{\sigma}}$$

$\int \bullet \, dU$: *integral over the Haar measure on the full unitary group over* $\mathcal{H}_A$

$$\sigma_E = \mathrm{Tr}_{A'}\sigma_{A'E}$$

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- **One-shot decoupling theorem:** *[Dupuis et al]*

Let $\varepsilon > 0$; $\tilde{\mathcal{N}}^{A\to E}$; $\sigma_{A'E} = (\mathrm{id}_{A'} \otimes \tilde{\mathcal{N}})\Phi_{A'A}$

Then for any state ρ_{RA}, *Choi state of* $\tilde{\mathcal{N}}$

$$\int \|\omega_{RE}(U) - \rho_R \otimes \sigma_E\|_1 \, dU \leq 2^{-\frac{1}{2}H^{\varepsilon}_{\min}(A|R)_\rho - \frac{1}{2}H^{\varepsilon}_{\min}(A'|E)_\sigma}$$

ε

$$\omega_{RE}(U) = \omega_{RE} = (\mathrm{id} \otimes \tilde{\mathcal{N}})\Big((I \otimes U)\rho_{RA}(I \otimes U^\dagger)\Big)$$

$\int \bullet \, dU$: *integral over the Haar measure on the full unitary group over* $\mathcal{H}_A$

$$\sigma_E = \mathrm{Tr}_{A'}\sigma_{A'E}$$

- **One-shot decoupling theorem:** *[Dupuis et al]*

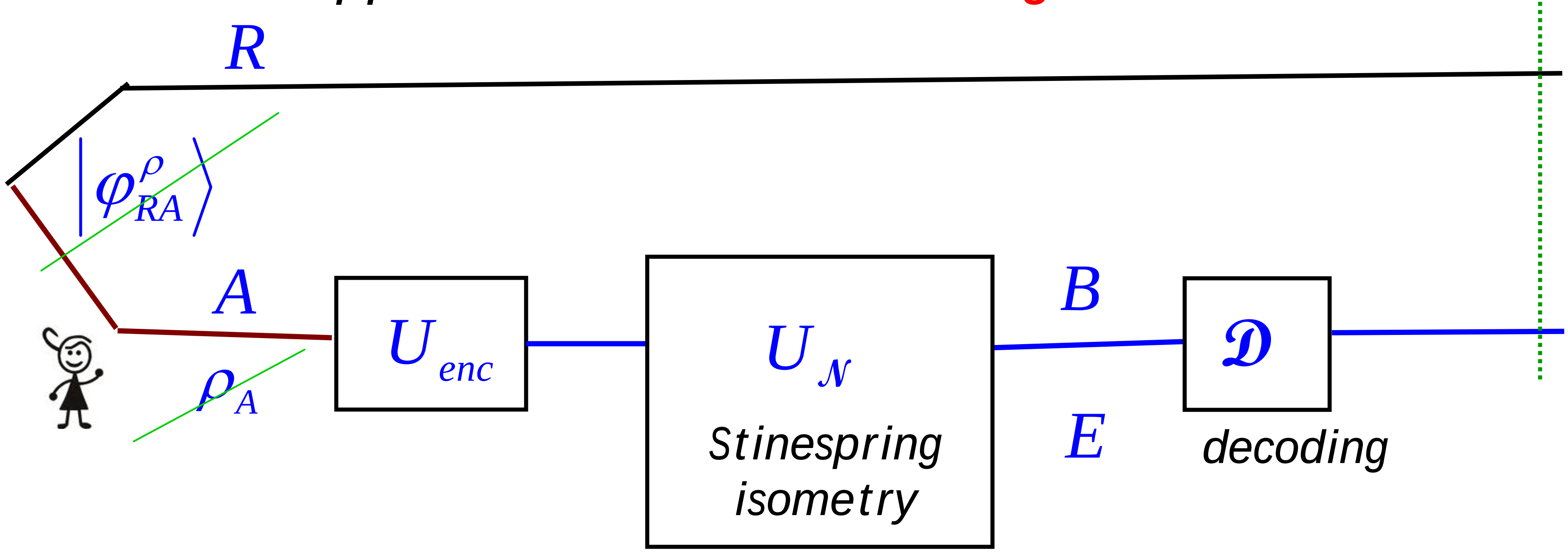
implies that: $\exists U :$

$$\| \omega_{RE}(U) - \rho_R \otimes \sigma_E \|_1 \leq^{\varepsilon} 2^{-\frac{1}{2} H_{\min}^{\varepsilon}(A|R)_\rho - \frac{1}{2} H_{\min}^{\varepsilon}(A'|E)_\sigma}$$

$$\omega_{RE}(U) = \omega_{RE} = (\mathrm{id} \otimes \tilde{\mathcal{N}})\Big((I \otimes U)\rho_{RA}(I \otimes U^\dagger)\Big)$$

$$\sigma_{A'E} = (\mathrm{id}_{A'} \otimes \tilde{\mathcal{N}})\Phi_{A'A}$$ Choi state of $\tilde{\mathcal{N}}$

Application: one-shot entanglement transmission



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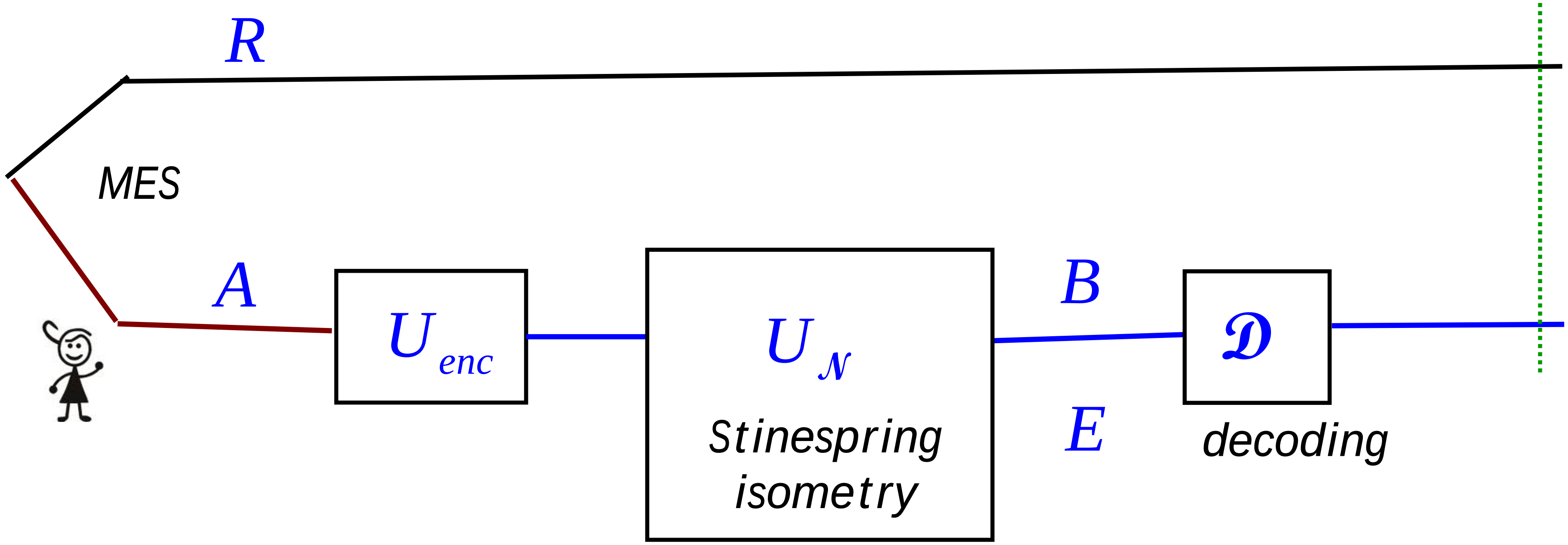
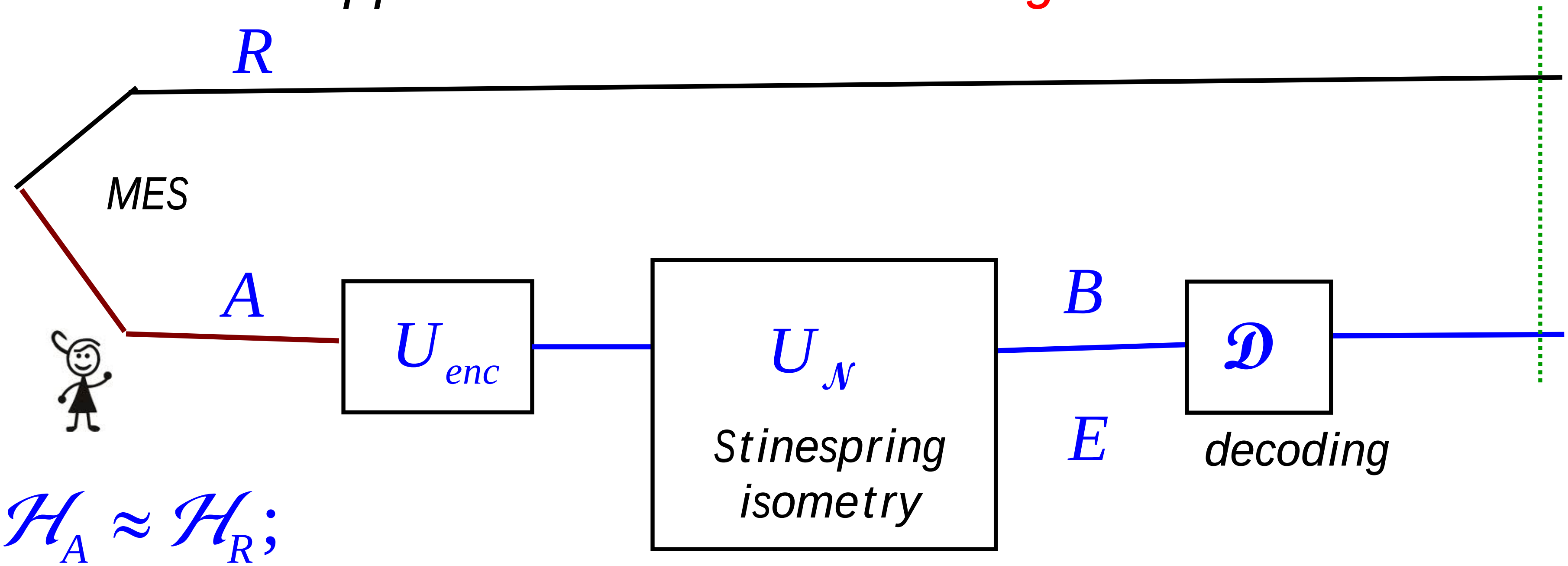
Application: one-shot entanglement transmission



Application: one-shot entanglement transmission



$\mathcal{H}_A \approx \mathcal{H}_R;$

- Alice locally prepares a maximally entangled state;
- A, R are both in her possession

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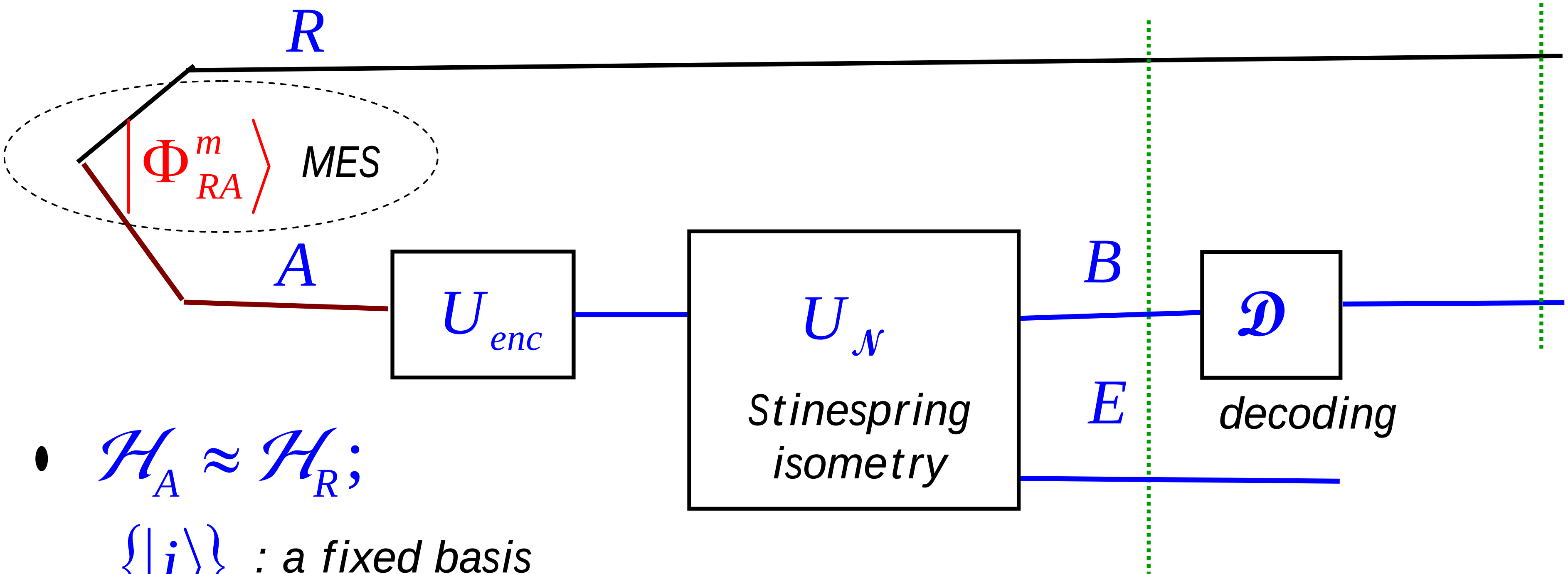
Application: one-shot entanglement transmission

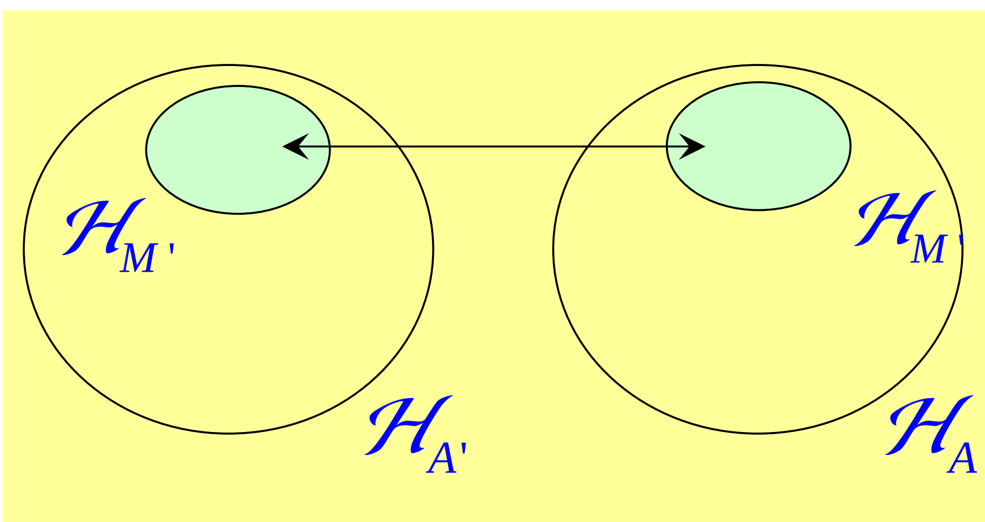


- $\mathcal{H}_A \approx \mathcal{H}_R$;

  $\{|i\rangle\}$: a fixed basis

$$|\Phi^m_{RA}\rangle = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} |i\rangle|i\rangle$$

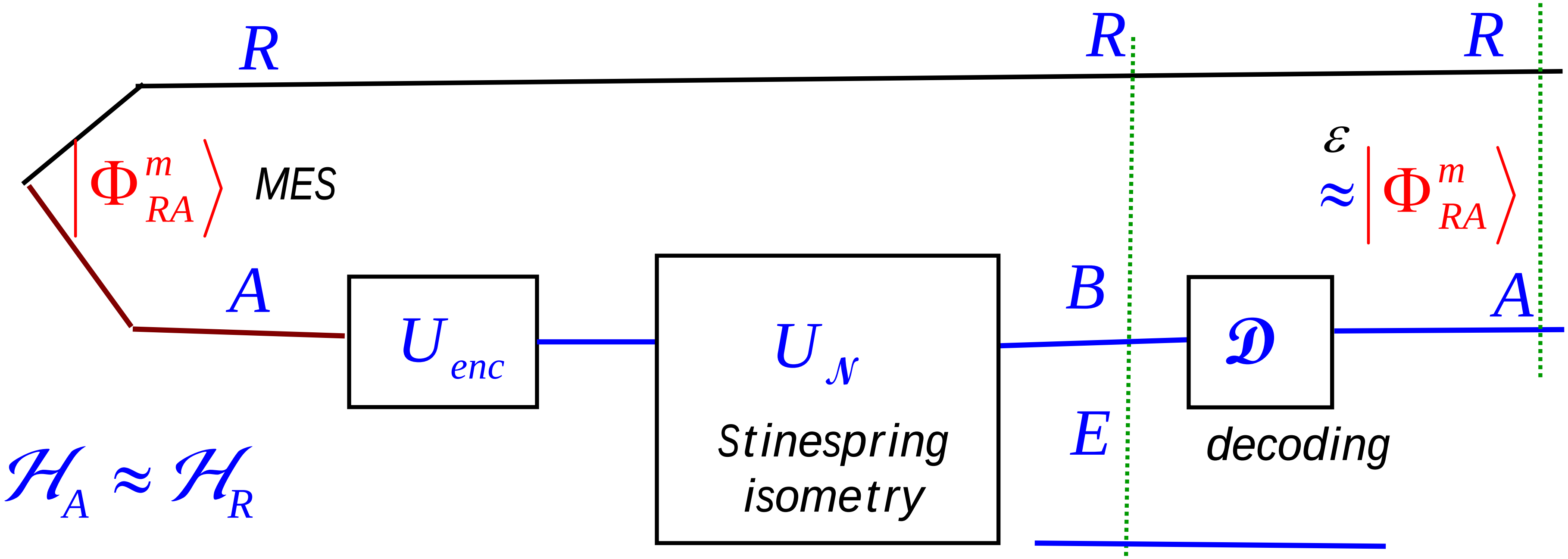
(a MES) $\in \mathcal{H}_{M'} \otimes \mathcal{H}_M$



$\mathcal{H}_M \approx \mathcal{H}_{M'}$;

$m = \dim \mathcal{H}_M$

Application: one-shot entanglement transmission



$$|\Phi^m_{RA}\rangle \in \mathcal{H}_{M'} \otimes \mathcal{H}_M \subseteq \mathcal{H}_R \otimes \mathcal{H}_A$$

- *Alice prepares a MES* $|\Phi^m_{RA}\rangle$; both systems R & A are with her
- *Aim: to transmit the system A to Bob*
- *such that – after decoding, the state that Bob shares with Alice is* $\varepsilon-$ *close to* $|\Phi^m_{RA}\rangle$

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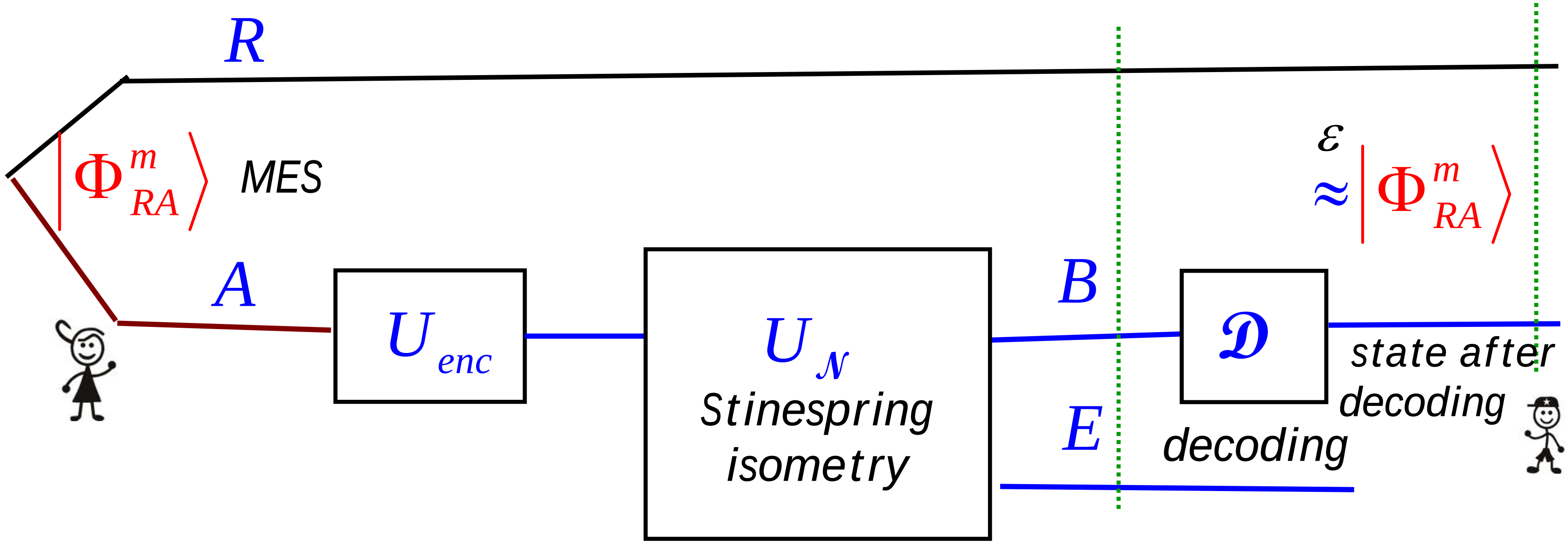
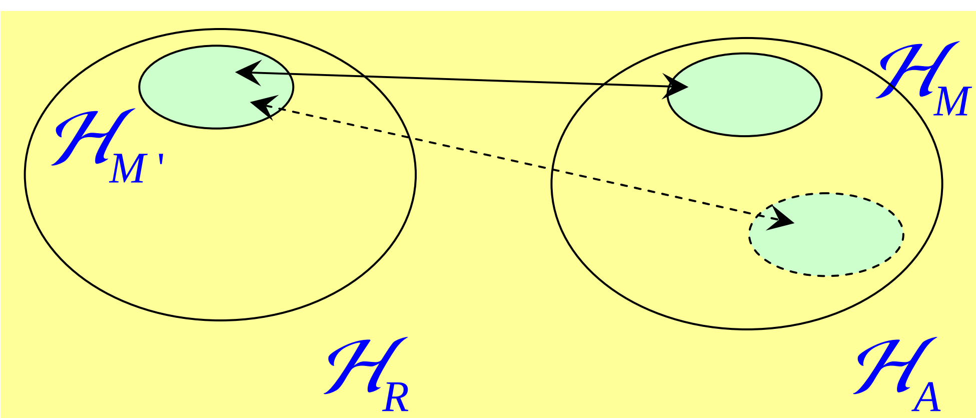
Application: one-shot entanglement transmission

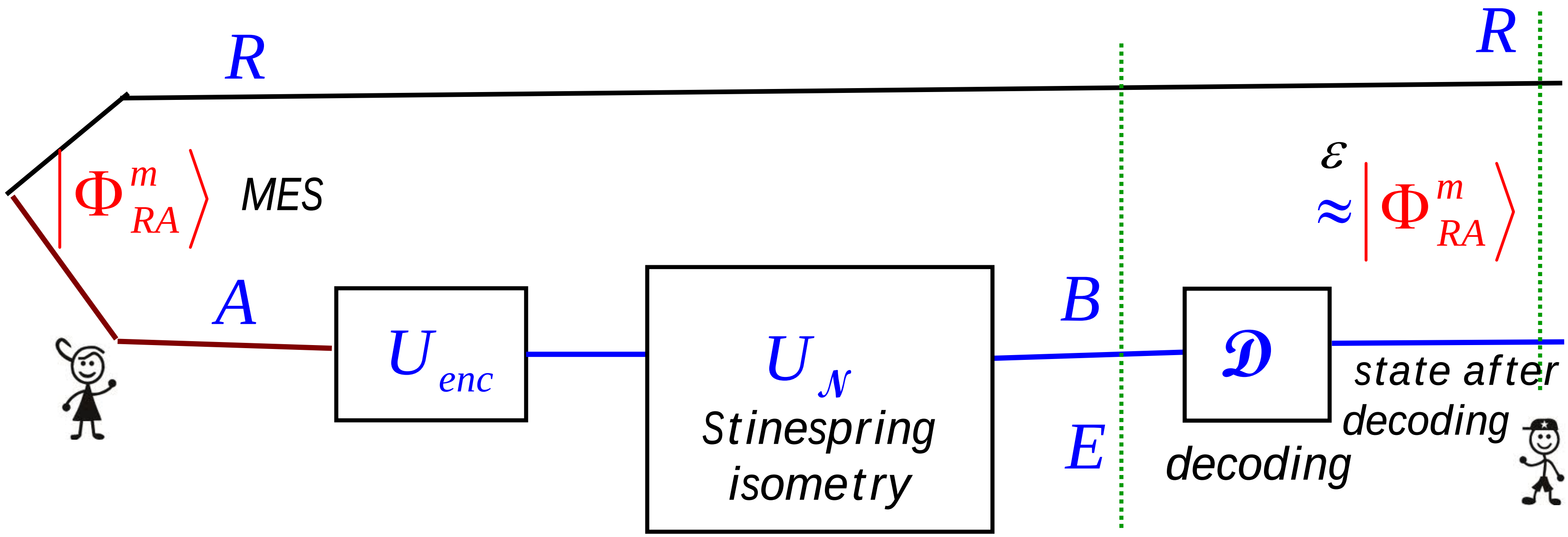


Role of the encoding map: U_{enc}

To select a suitable coding subspace which is almost error-free



Application: one-shot entanglement transmission



$|\Phi^m_{RA}\rangle$ = MES of Schmidt rank m,

number of ebits transmitted (up to error ε) $= \log m$

- *Capacity* $:=$ *maximum number of ebits transmitted*

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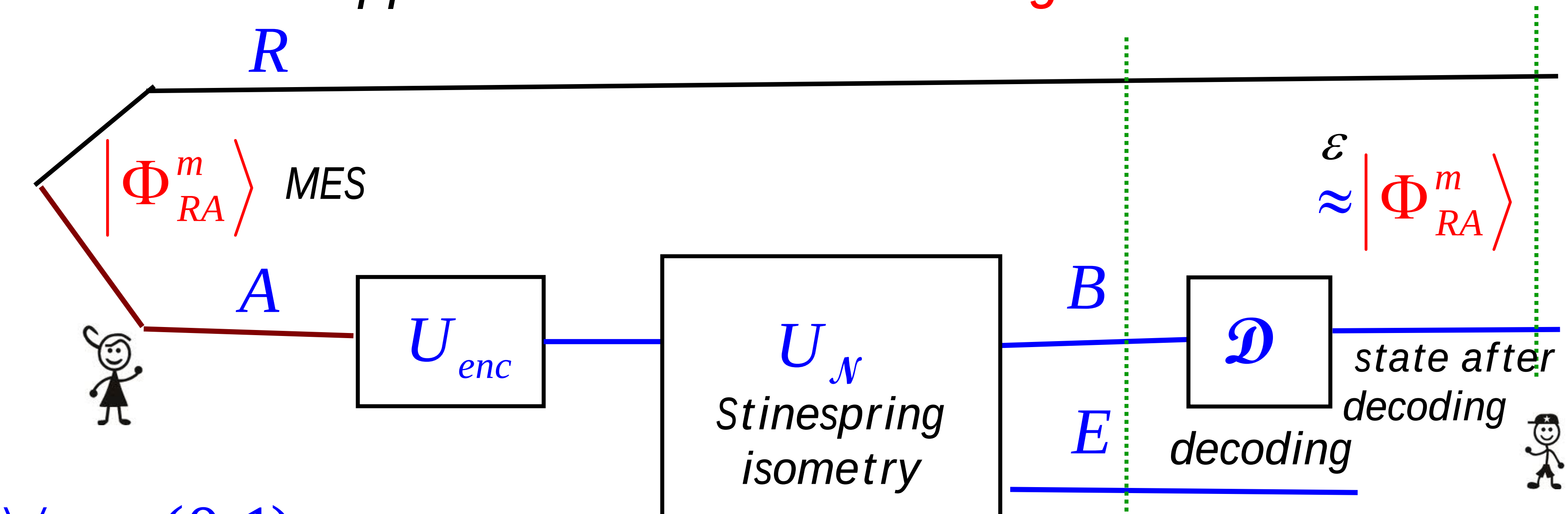
Application: one-shot entanglement transmission



$\forall \varepsilon \in (0,1):$

One-shot $\varepsilon -$ error entanglement-transmission capacity:

$$Q_{et}^{(1),\varepsilon}(\mathcal{N}) := \sup\left\{ \log m : \text{ final state } \overset{\varepsilon}{\approx} \Phi^m_{RA} \right\}$$

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Application: one-shot entanglement transmission

- *Theorem: [ND, M-H.Hsieh; F.Buscemi & ND]*

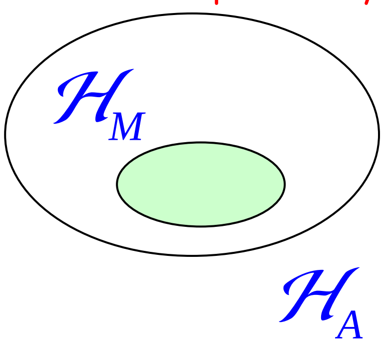
One-shot $\varepsilon -$ *error entanglement-transmission capacity,*

$\forall \varepsilon \in (0,1):$

$$Q_{et}^{(1),\varepsilon}(\mathcal{N}) \approx \max_{\mathcal{H}_M \subseteq \mathcal{H}_A} \left\{ -H_{\max}^{\varepsilon}(R|B)_{\sigma} \right\}$$

$$\sigma_{RB} = (\mathrm{id}_R \otimes \mathcal{N}^{A\to B})\Phi_{RA}^{m};$$

Since $|\Phi_{RA}^{m}\rangle \in \mathcal{H}_{M'} \otimes \mathcal{H}_M$ *, action of* $\mathcal{N}$ *restricted to* $\mathcal{H}_M$



- σ_{RB} *depends on the choice of* $\mathcal{H}_M$
- *hence maximise over all* $\mathcal{H}_M \subseteq \mathcal{H}_A$

Application: one-shot entanglement transmission

- *Theorem: [ND, M-H.Hsieh; F.Buscemi & ND]*

One-shot $\varepsilon -$ *error entanglement-transmission capacity,*

$\forall \varepsilon \in (0,1):$

$$Q_{et}^{(1),\varepsilon}(\mathcal{N}) \approx \max_{\mathcal{H}_M \subseteq \mathcal{H}_A} \left\{ -H_{\max}^{\varepsilon}(R|B)_\sigma \right\}$$

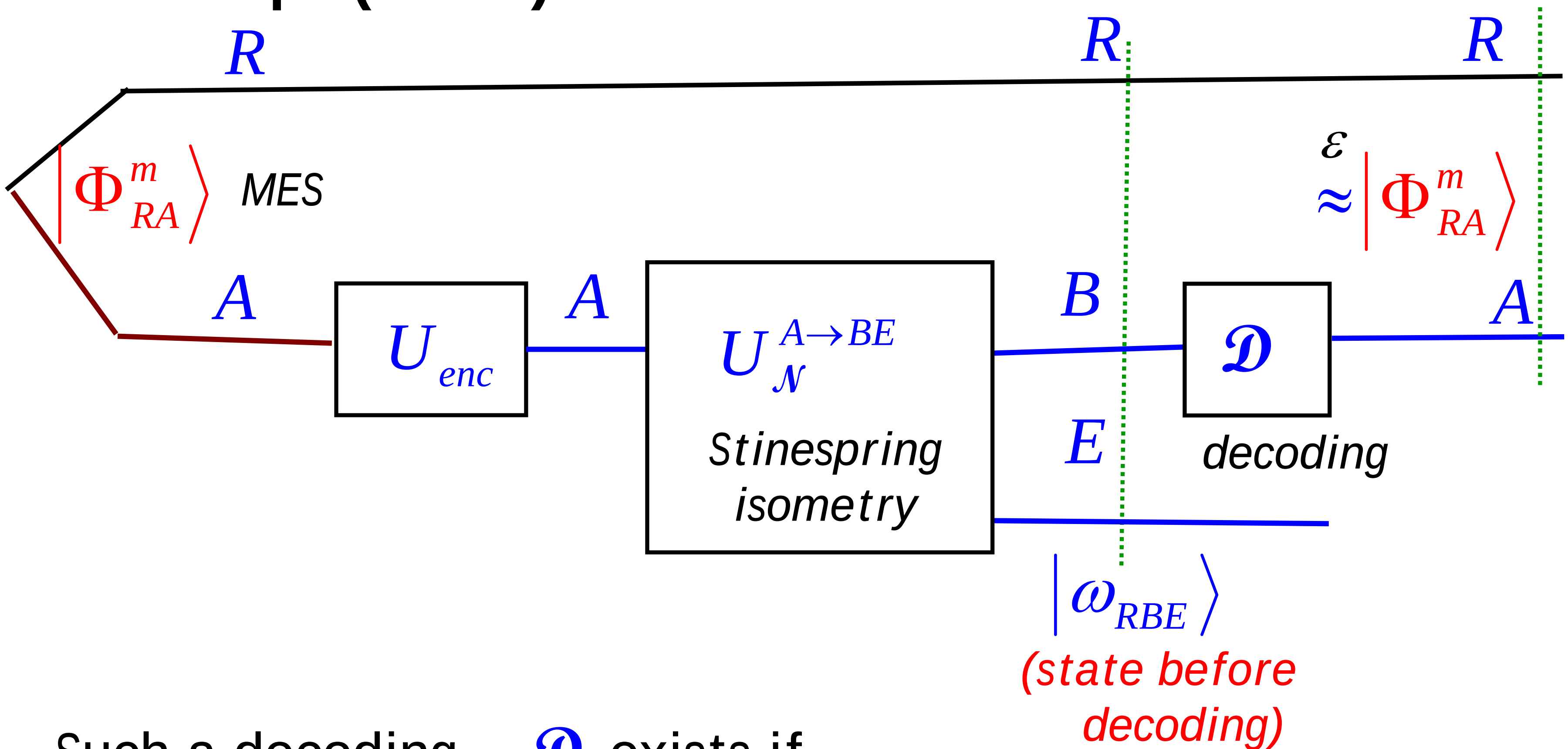
$$\max_{\mathcal{H}_M \subset \mathcal{H}_A} \left\{ -H_{\max}^{\varepsilon}(R|B)_\sigma \right\} + f(\varepsilon) \le Q_{et}^{(1),\varepsilon}(\mathcal{N})$$

$$\le \max_{\mathcal{H}_M \subset \mathcal{H}_A} \left\{ -H_{\max}^{\varepsilon}(R|B)_\sigma \right\}$$

- **Proof:** **Step I (Achievability) ;** *lower bound* **Step II (Converse)** *upper bound*

Application: one-shot entanglement transmission

- **Proof: Step I (Direct)** *lower bound*



- Such a decoding $\mathcal{D}$ exists if

$$\omega_{RE} \overset{\varepsilon}{\approx} \rho_R \otimes \sigma_E; \qquad \rho_R = \frac{I}{m}$$ *(completely mixed state)*

- Use one-shot decoupling theorem

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- **One-shot decoupling theorem:** *[Dupuis et al]*

implies that: $\exists U:$

$$\| \omega_{RE}(U) - \rho_R \otimes \sigma_E \|_1^{\varepsilon} \leq 2^{-\frac{1}{2} H_{\min}^{\varepsilon}(A|R)_{\Phi^m} - \frac{1}{2} H_{\min}^{\varepsilon}(R|E)_{\sigma}}$$

$$\omega_{RE}(U) = \omega_{RE} = (\mathrm{id} \otimes \tilde{\mathcal{N}})\big((I \otimes U)\Phi_{RA}^m (I \otimes U^\dagger)\big)$$

$$\sigma_{RE} = (\mathrm{id}_R \otimes \tilde{\mathcal{N}})\Phi_{RA}^m; \text{ Choi state of } \tilde{\mathcal{N}}$$

$$\because \rho_{RA} = \Phi_{RA}^m$$

- *since action of* $\mathcal{N}$ ($\&\ \therefore \tilde{\mathcal{N}}$) *restricted to* $\mathcal{H}_M \subseteq \mathcal{H}_A$
- *and* Φ_{RA}^m *is a MES in* $\mathcal{H}_{M'} \otimes \mathcal{H}_M$

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- **One-shot decoupling theorem:** *[Dupuis et al]*

implies that: $\exists U:$

$$\| \omega_{RE}(U) - \rho_R \otimes \sigma_E \|_1 \overset{\varepsilon}{\leq} 2^{-\frac{1}{2}H_{\min}^{\varepsilon}(A|R)_{\Phi^m} - \frac{1}{2}H_{\min}^{\varepsilon}(R|E)_{\sigma}}$$ ......(a)

$$\omega_{RE}(U) = \omega_{RE} = \; = (\mathrm{id} \otimes \tilde{\mathcal{N}})\Big((I \otimes U)\Phi_{RA}^m (I \otimes U^\dagger)\Big)$$

$$\sigma_{RE} = (\mathrm{id}_R \otimes \tilde{\mathcal{N}})\Phi_{RA}^m;$$

$$\because \rho_{RA} = \Phi_{RA}^m$$

- Require : **RHS** of (a) to be small

$$\Rightarrow \| \omega_{RE} - \rho_R \otimes \sigma_E \|_1 \overset{\varepsilon}{\approx} 0 \Rightarrow \omega_{RE} \overset{\varepsilon}{\approx} \rho_R \otimes \sigma_E$$

(approx.) **decoupling!**

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- **One-shot decoupling theorem:** *[Dupuis et al]*

implies that: $\exists U:$

$$\| \omega_{RE}(U) - \rho_R \otimes \sigma_E \|_1 \leq 2^{-\frac{1}{2} H_{\min}^{\varepsilon}(A|R)_{\Phi^m} - \frac{1}{2} H_{\min}^{\varepsilon}(R|E)_{\sigma}} \quad \ldots\ldots\text{(a)}$$

ε

Note: $H_{\min}^{\varepsilon}(A|R)_{\Phi^m} \geq H_{\min}(A|R)_{\Phi^m} = -\log m$

$$\sigma_{RE} = (\mathrm{id}_R \otimes \tilde{\mathcal{N}})\Phi_{RA}^m;$$

Purification: $|\sigma_{RBE}\rangle = (\mathrm{id}_R \otimes U_{\mathcal{N}}^{A\to BE})\Phi_{RA}^m$

Duality of smoothed min- and max- entropies: $H_{\min}^{\varepsilon}(R|E)_{\sigma} = -H_{\max}^{\varepsilon}(R|B)_{\sigma}$

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$$\| \omega_{RE} - \rho_R \otimes \sigma_E \|_1 \overset{\varepsilon}{\leq} 2^{-\frac{1}{2} H^{\varepsilon}_{\min}(A|R)_{\Phi^m} - \frac{1}{2} H^{\varepsilon}_{\min}(R|E)_{\sigma}}$$

$$\| \omega_{RE} - \rho_R \otimes \sigma_E \|_1 \overset{\varepsilon}{\leq} 2^{\frac{1}{2}\log m + \frac{1}{2} H^{\varepsilon}_{\max}(R|B)_{\sigma}} \qquad \ldots\ldots(a)$$

- **Decoupling occurs if:** RHS of (a) is small :

$$\log m = -H^{\varepsilon}_{\max}(R \mid B)_{\sigma} + \log \varepsilon;$$

- **One-shot entanglement transmission capacity:**

$$Q^{(1),\varepsilon}_{et}(\mathcal{N}) \geq \max_{\mathcal{H}_M \subseteq \mathcal{H}_A} \left\{ -H^{\varepsilon}_{\max}(R \mid B)_{\sigma} \right\} + \log \varepsilon$$

■

In summary

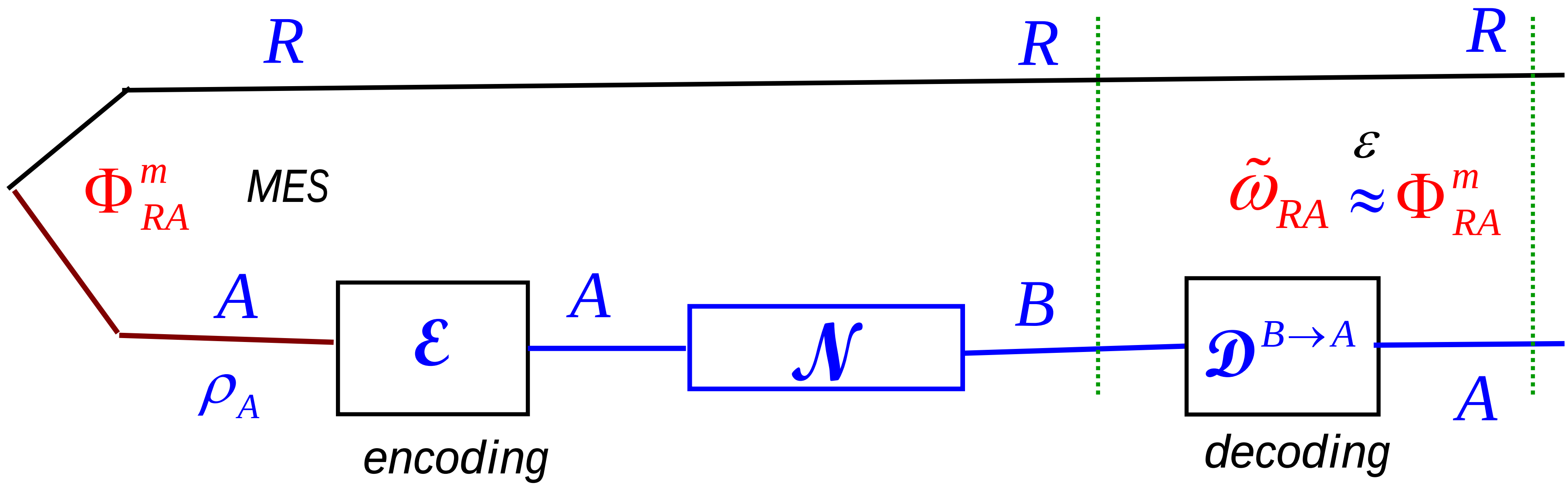
- *We established the lower bound:*

$$Q_{et}^{(1),\varepsilon}(\mathcal{N}) \geq \max_{\mathcal{H}_M \subseteq \mathcal{H}_A} \left\{ -H_{\max}^{\varepsilon}(R|B)_{\sigma} \right\} + f(\varepsilon)$$

- *Used the fact: decoupling* $\Rightarrow \exists$ *a decoder*
- *Condition for decoupling* → *lower bound*

Converse: $Q_{et}^{(1),\varepsilon}(\mathcal{N}) \le \max_{\mathcal{H}_M \subseteq \mathcal{H}_A} \left\{ -H_{\max}^{\varepsilon}(R|B)_\sigma \right\}$

- **Proof:**
  - Assume that $\exists$ an encoding $\mathcal{E}$ & a decoding $\mathcal{D}$ such that



$\log m \le ?$

$\Rightarrow Q_{et}^{(1),\varepsilon}(\mathcal{N}) \le ?$

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- **Converse:** $Q_{et}^{(1),\varepsilon}(\mathcal{N}) \le \max_{\mathcal{H}_M \subseteq \mathcal{H}_A} \left\{ -H_{\max}^{\varepsilon}(R|B)_{\sigma} \right\}$

$$\log m = -H_{\max}(R|A)_{\Phi_{RA}^m}$$

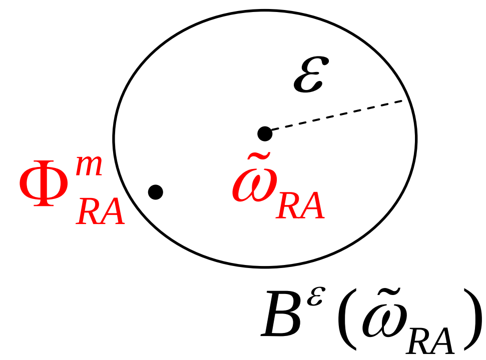
$$\le \max_{\zeta_{RA} \in B^{\varepsilon}(\tilde{\omega}_{RA})} \left\{ -H_{\max}(R|A)_{\zeta} \right\}$$

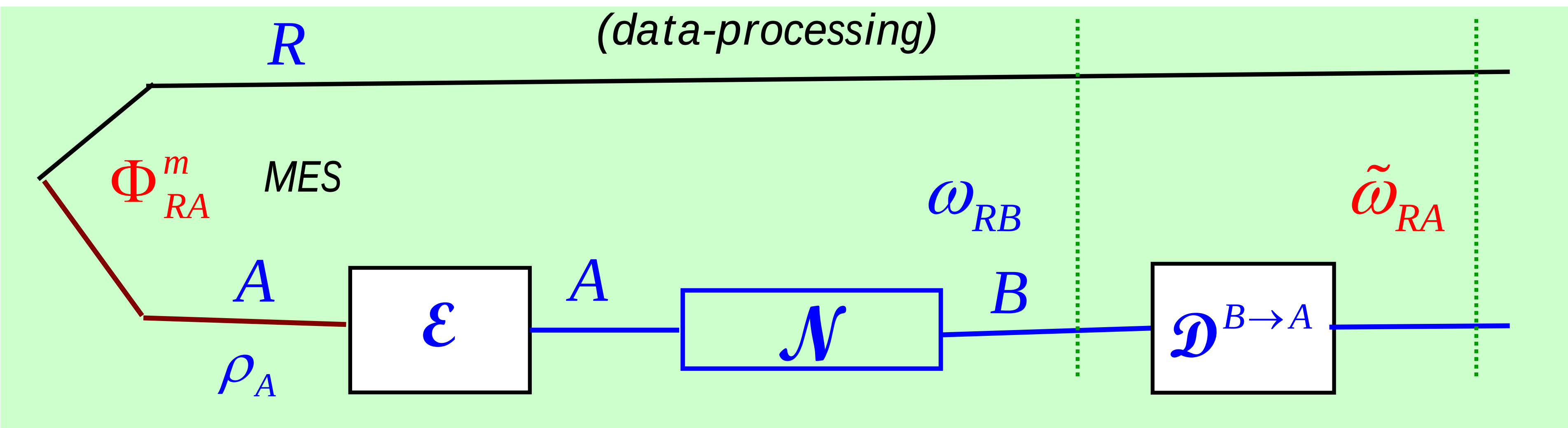
$$= -\min_{\zeta_{RA} \in B^{\varepsilon}(\tilde{\omega}_{RA})} \left\{ H_{\max}(R|A)_{\zeta} \right\}$$

$$= -H_{\max}^{\varepsilon}(R|A)_{\tilde{\omega}} \le -H_{\max}^{\varepsilon}(R|B)_{\omega} \quad \because \mathcal{D}^{B\to A}$$

State after decoding :

$$\tilde{\omega}_{RA} \overset{\varepsilon}{\approx} \Phi_{RA}^m$$





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- **Converse:** $Q_{et}^{(1),\varepsilon}(\mathcal{N}) \le \max_{\mathcal{H}_M \subseteq \mathcal{H}_A} \left\{ -H_{\max}^{\varepsilon}(R|B)_{\sigma} \right\}$

Thus:

$$\log m \le -H_{\max}^{\varepsilon}(R|B)_{\omega}$$

where

$$\omega_{RB} = (\mathrm{id}_R \otimes \mathcal{N}^{A\to B} \circ \mathcal{E})\Phi_{RA}^{m};$$

$$* \Downarrow$$

$$\le -H_{\max}^{\varepsilon}(R|B)_{\sigma}$$

where

$$\sigma_{RB} = (\mathrm{id}_R \otimes \mathcal{N}^{A\to B})\Phi_{RA}^{m};$$

Main ingredients of * :

- *Ricochet:* $(I \otimes A)|\Phi\rangle = (A^T \otimes I)|\Phi\rangle$

- *Invariance of smooth conditional max-entropy under local unitaries*

$$H_{\max}^{\varepsilon}(R|B)_{\omega} = H_{\max}^{\varepsilon}(R|B)_{\sigma} \quad \text{if} \quad \omega_{RB} \overset{U}{\leftrightarrow} \sigma_{RB}$$

$$\Rightarrow \log m \le -H_{\max}^{\varepsilon}(R|B)_{\sigma}$$

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- **Converse:** $Q_{et}^{(1),\varepsilon}(\mathcal{N}) \le \max_{\mathcal{H}_M \subseteq \mathcal{H}_A} \{-H_{\max}^{\varepsilon}(R|B)_\sigma\}$

$$\log m \le -H_{\max}^{\varepsilon}(R|B)_\omega$$

$$\sigma_{RB} = (\mathrm{id}_R \otimes \mathcal{N}^{A\to B})\Phi_{RA}^m;$$

$$\omega_{RB} = (\mathrm{id}_R \otimes \mathcal{N}^{A\to B} \circ \mathcal{E})\Phi_{RA}^m;$$

$$= (\mathrm{id}_R \otimes \mathcal{N}^{A\to B})(\mathrm{id}_R \otimes \mathcal{E})\Phi_{RA}^m;$$

$$= (\mathrm{id}_R \otimes \mathcal{N}^{A\to B})(\mathcal{E}^T \otimes \mathrm{id}_A)\Phi_{RA}^m;$$

$$= (\mathcal{E}^T \otimes \mathrm{id}_A)(\mathrm{id}_R \otimes \mathcal{N}^{A\to B})\Phi_{RA}^m;$$

$$\omega_{RB} = (\mathcal{E}^T \otimes \mathrm{id}_A)\sigma_{RB}$$

Data-processing inequality

$$H_{\max}^{\varepsilon}(R|B)_\sigma \le H_{\max}^{\varepsilon}(R|B)_\omega$$

$$\log m \le -H_{\max}^{\varepsilon}(R|B)_\omega \le -H_{\max}^{\varepsilon}(R|B)_\omega$$

$$\Rightarrow \log m \le -H^{\varepsilon}_{\max}(R|B)_{\sigma}$$

$$Q^{(1),\varepsilon}_{et}(\mathcal{N}) \le \max_{\mathcal{H}_M \subseteq \mathcal{H}_A} \left\{-H^{\varepsilon}_{\max}(R|B)_{\sigma}\right\}$$

■

In summary

- *We established the upper bound (converse) by starting with the assumption that :*

  $\exists$ *an encoding* $\mathcal{E}$ *& a decoding* $\mathcal{D}$ *such that*

  decoded state is $\stackrel{\varepsilon}{\approx} \Phi^m_{RA}$

- *Going from* $\log m = -H_{\max}(R|A)_{\Phi^m_{RA}}$ → *smooth conditional max-entropy*
- *Using data-processing inequality*
- *& invariance of smooth conditional max-entropy under unitaries*

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One-shot $\varepsilon-$ *error* *entanglement-transmission* *capacity:*

$$Q_{et}^{(1),\varepsilon}(\mathcal{N}) \approx \max_{\mathcal{H}_M \subseteq \mathcal{H}_A} \left\{ -H_{\max}^{\varepsilon}(R\,|\,B)_{\sigma} \right\}$$

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One-shot setting → **Asymptotic memoryless setting**

Asymptotic capacity

$$Q_{et}(\mathcal{N}) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} Q_{et}^{(1),\varepsilon}(\mathcal{N}^{\otimes n})$$

One-shot result:

$$Q_{et}^{(1),\varepsilon}(\mathcal{N}) \approx \max_{\mathcal{H}_M \subseteq \mathcal{H}_A} \left\{ -H_{\max}^{\varepsilon}(R \mid B)_{\sigma} \right\}$$

$$\ldots. \leq Q_{et}^{(1),\varepsilon}(\mathcal{N}^{\otimes n}) \leq \ldots.$$

One-shot setting → Asymptotic memoryless setting

Asymptotic capacity

$$Q_{et}(\mathcal{N}) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} Q_{et}^{(1),\varepsilon}(\mathcal{N}^{\otimes n})$$

One-shot result:

$$Q_{et}^{(1),\varepsilon}(\mathcal{N}) \approx \max_{\mathcal{H}_M \subseteq \mathcal{H}_A} \left\{ -H_{\max}^{\varepsilon}(R|B)_{\sigma} \right\}$$

$$\ldots \le Q_{et}^{(1),\varepsilon}(\mathcal{N}^{\otimes n}) \le \max_{\mathcal{H}_{M_n} \subseteq \mathcal{H}_A^{\otimes n}} \left\{ -H_{\max}^{\varepsilon}(R|B)_{\sigma_n} \right\}$$

$n \to \infty$

$$Q_{et}(\mathcal{N}) = \lim_{n \to \infty} \frac{1}{n} \max_{\mathcal{H}_{M_n} \subseteq \mathcal{H}_A^{\otimes n}} \left\{ -S(R|B)_{\sigma_n} \right\}$$

$$\sigma_n = \sigma_{R_n B_n} = (\mathrm{id}_{R_n} \otimes \mathcal{N}^{\otimes n}) \Phi_{R_n A_n}^{m_n}$$

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One-shot setting → Asymptotic memoryless setting

$$Q_{et}(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} \max_{\mathcal{H}_{M_n} \subset \mathcal{H}_A^{\otimes n}} \left\{ -S(R\,|\,B)_{\sigma_n} \right\}$$

$$I_{\sigma_n}^{R>B} = -S(\sigma_{RB}) + S(\sigma_B) = -S(R\,|\,B)_{\sigma_n}$$

coherent information

Entanglement transmission capacity (in asymptotic, memoryless setting)

$$Q_{et}(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} \max_{\mathcal{H}_{M_n} \subset \mathcal{H}_A^{\otimes n}} I_{\sigma_n}^{R_n > B_n}$$ *[Lloyd, Shor, Devetak]*

regularized coherent information

$$\sigma_n \equiv \sigma_{R_n B_n} = (\mathrm{id}_{R_n} \otimes \mathcal{N}^{\otimes n}) \Phi_{R_n A_n}^{m_n}$$

Summary

- Quantum information transmission through a noisy quantum channel $\mathcal{N}$ in the one-shot setting

- Decoupling → existence of a decoder:

such that Bob can recover the quantum state sent by Alice up to an error ε

- One-shot entanglement transmission through a quantum channel : bounds on the capacity

-- given in terms of the smooth conditional max-entropy

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Summary contd.

- One-shot entanglement transmission capacity

$$Q_{et}^{(1),\varepsilon}(\mathcal{N}) \approx \max_{\mathcal{H}_M \subseteq \mathcal{H}_A} \left\{ -H_{\max}^{\varepsilon}(R|B)_{\sigma} \right\}$$

- This yields bounds on the one-shot quantum capacity of $\mathcal{N}$

since

$$Q_{et}^{(1),\frac{\varepsilon}{2}}(\mathcal{N}) \leq Q^{(1),\varepsilon}(\mathcal{N}) \leq Q_{et}^{(1),\varepsilon}(\mathcal{N})$$

one-shot quantum capacity

- Retrieve known asymptotic result of Lloyd, Shor & Devetak:

-- given in terms of the regularized coherent information

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Optimal rates of Info-processing tasks

One-shot setting

$(n < \infty)$

given in terms of smoothed entropies

obtained from:

$D_{\min}(\rho \| \sigma),\ D_{\max}(\rho \| \sigma), D_0(\rho \| \sigma)$

Asymp. memoryless settng

$(n \to \infty)$

given in terms of entropies

obtained from:

$D(\rho \| \sigma)$

Quantum Asymptotic Equipartition Property

- e.g

$$\forall \varepsilon > 0, \quad \limsup_{n\to\infty} \frac{1}{n} D^{\varepsilon}_{\max}(\rho^{\otimes n} \| \sigma^{\otimes n}) \equiv D(\rho \| \sigma)$$

One-shot bounds $\xrightarrow{n \to \infty}$ *asymptotic, i.i.d. result*